On Subspaces of Non-commutative L_p-Spaces

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Abstract: We study some structural aspects of the subspaces of the non-commutative (Haagerup) L_p-spaces associated with a general (non necessarily semi-finite) von Neumann algebra $\mathcal{A}$. If a subspace X of $L_p(\mathcal{A})$ contains uniformly the spaces ℓ_p^n, $n \geq 1$, it contains an almost isometric, almost 1-complemented copy of ℓ_p. If X contains uniformly the finite dimensional Schatten classes S_p^n, it contains their ℓ_p-direct sum too. We obtain a version of the classical Kadec-Pełczyński dichotomy theorem for L_p-spaces, $p \geq 2$. We also give operator space versions of these results. The proofs are based on previous structural results on the ultrapowers of $L_p(\mathcal{A})$, together with a careful analysis of the elements of an ultrapower $L_p(\mathcal{A})_{\mathcal{U}}$ which are disjoint from the subspace $L_p(\mathcal{A})$. These techniques permit to recover a recent result of N. Randrianantoanina concerning a Subsequence Splitting Lemma for the general non-commutative L_p spaces. Various notions of p-equiintegrability are studied (one of which is equivalent to Randrianantoanina's one) and some results obtained by Haagerup, Rosenthal and Sukochev for L_p-spaces based on finite von Neumann algebras concerning subspaces of $L_p(\mathcal{A})$ containing ℓ_p are extended to the general case.

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0. Introduction

Since several years the study of non-commutative L_p-spaces has incited new interest because of their close relations with the new and rapidly developing Operator Space Theory and Non-commutative Probability Theory. It is known now that non-commutative integration is a fundamental tool in both latter theories. Conversely, results and problems from these theories permit to gain new insight into the theory of non-commutative L_p-spaces and at the same time pose new problems in the frame of this theory: see for instance the recent works [EJR], [Ju1-3], [JX], [NO], [O], [PX], [R1-4].

The starting point of the present work is a problem arising from the theory of $\mathcal{OL}_p$-spaces, which was initiated by Effros and Ruan ([ER1]) and developed in the recent paper [JNRX] (see also [JOR], [NO]). To explain this, we first recall the famous Kadec-Pełczyński dichotomy theorem, which states that every closed subspace of $L_p(0,1)$, $2 < p < \infty$, either is isomorphic to a Hilbert space or contains a subspace which is isomorphic to ℓ_p and complemented in $L_p(0,1)$. This theorem plays an important role in the classical theory of $\mathcal{L}_p$-spaces. The class of $\mathcal{OL}_p$-spaces is an analog for the category of operator spaces of the class of $\mathcal{L}_p$-spaces in the category of Banach spaces; when going back to the Banach space category by forgetting the matricial structure, the class of $\mathcal{OL}_p$-spaces gives rise to a still new class of Banach spaces, which could be called "non-commutative $\mathcal{L}_p$-spaces": X belongs to this class if for some λ, every finite dimensional subspace of X is contained in another subspace, which is λ-isomorphic to a finite dimensional non-commutative L_p-space. It is then natural to look for a non-commutative version of the Kadec-Pełczyński dichotomy. A version of the usual Kadec-Pełczyński's dichotomy in a non-commutative setting exists already in the literature, with exactly the same statement; it was proved indeed in [S] for non-commutative L_p-spaces based on finite von Neumann algebras (under an equivalent form), and in [R2] for semi-finite ones.

This version however does not help at all in developing the theory of $\mathcal{OL}_p$-spaces (the conclusion it gives is "too commutative" in a certain sense). A stronger, still very hypothetical statement would be preferable in this direction:

A closed subspace of a non-commutative L_p-space ($2 < p < \infty$) should either be embeddable into a <u>commutative</u> L_p-space or contain a copy of the p-direct sum $K_p = (\bigoplus_{n\geq 1} S_p^n)_p$ of the finite dimensional p-Schatten classes.

A step towards this direction is made in the present paper, namely the following theorem, which is one of our main results:

Theorem 0.1. *Let $\mathcal{A}$ be a von Neumann algebra (non necessarily semi-finite), $0 < p < \infty$, $p \neq 2$ and X a closed subspace of $L_p(\mathcal{A})$. Assume that for some constant $\lambda \geq 1$, and for every $n \geq 1$, X contains a subspace λ-isomorphic to the space S_p^n (resp. and μ-complemented in $L_p(\mathcal{A})$ – in this case we suppose $p \geq 1$). Then for every $\varepsilon > 0$, X contains a subspace $(\lambda+\varepsilon)$-isomorphic to K_p (resp. and $(\lambda\mu+\varepsilon)$-complemented in $L_p(\mathcal{A})$).*

This result has a forerunner in the case $\mathcal{A} = B(\ell_2)$ (then $L_p(\mathcal{A})$ is the usual Schatten class S_p), which was obtained by Arazy and Lindenstrauss in [ArL]. Their proof, which relies on a careful analysis of the local structure of S_p together with a clever use of Ramsey's theorem, can be extended to some special cases of Theorem 0.1 (e.g. when $\mathcal{A}$ is finite and $p > 2$) but we hardly imagine how to adapt it to the general situation described in Theorem 0.1. Our proof of Theorem 0.1 heavily depends on ultrapower techniques, using the fact, proved in [Ra], that the class of non-commutative L_p-spaces is closed under ultrapowers.

In fact, we will see that the subspace of X isomorphic to K_p obtained in Theorem 0.1 is built over a sequence of subspaces isomorphic to the S_p^n's and "almost disjoint". This approach to Theorem 0.1 also allows us to extend to all von Neumann algebras the first non-commutative version of the Kadec-Pełczyński dichotomy mentioned previously, which remained an open question in the non semi-finite case. More precisely, we have:

Theorem 0.2. *Let $\mathcal{A}$ be a von Neumann algebra, $2 < p < \infty$ and X a closed subspace of $L_p(\mathcal{A})$. Then either X is isomorphic to a Hilbert space and complemented in $L_p(\mathcal{A})$ or X contains a subspace isomorphic to ℓ_p and complemented in $L_p(\mathcal{A})$.*

This paper is organized as follows. In section 1 we recall some necessary preliminaries on non-commutative L_p-spaces and their ultrapowers. The non-commutative L_p-spaces we consider are those constructed by Haagerup [H]. Contrary to the class of "usual" L_p-spaces associated with a normal faithful semi-finite trace, the class of Haagerup L_p-spaces is closed under ultraproducts ([Ra]). The main tools of the paper are developed in section 2, where we show how to push disjoint elements in an ultrapower of $L_p(\mathcal{A})$ down to disjoint elements of $L_p(\mathcal{A})$. Theorem 0.1 above will be proved in Section 3. In fact, we shall prove a more general result by replacing the spaces S_p^n by a sequence of finite dimensional spaces. Section 4 is devoted to the equiintegrability in $L_p(\mathcal{A})$. We give a rather complete study of the p-equiintegrable subsets of $L_p(\mathcal{A})$. Our techniques permit us to easily recover the Subsequence Splitting Lemma proved by N. Randrianantoanina [R3]. In section 5 we characterize the subspaces of $L_p(\mathcal{A})$ which contain a subspace isomorphic to ℓ_p. As a corollary, we get Theorem 0.2. Such characterizations are classical in the commutative case, and were recently proved for spaces associated with finite or semifinite von Neumann algebras in [HRS], [R1] and [SX]. The last section aims at extending the previous results to the operator space setting. There we get the operator space versions of Theorems 0.1 and 0.2. We also add an appendix whose result determines when equality occurs in the non-commutative Clarkson inequality. This result improves a previous theorem due to H. Kosaki [Ko2] and implies a characterization of isometric 2-dimensional ℓ_p-subspaces of $L_p(\mathcal{A})$ which is repeatedly used in the paper.

The main results of this paper were announced in the Note [RaX].

1. Preliminaries

This section contains notations, most notions and basic facts necessary to the whole paper. For clarity we divide it into three subsections.

Non-commutative L_p-spaces

There are several equivalent constructions of non-commutative L_p-spaces associated with a von Neumann algebra (c.f., e.g. [AM], [H], [Hi], [I], [Ko1], [Te2]). We shall use in this paper Haagerup's construction, which we recall briefly now (see [Te1] for a precise introduction to the subject). Let $\mathcal{A}$ be a von Neumann algebra. For $0 < p < \infty$, the spaces $L_p(\mathcal{A})$ are constructed as spaces of measurable operators relative not to $\mathcal{A}$ but to a certain semi-finite super von Neumann algebra of $\mathcal{A}$, namely, the crossed product of $\mathcal{A}$ by one of its modular automorphism groups. Let $\mathcal{M}$ be the crossed product of $\mathcal{A}$ by the modular automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ of a fixed normal faithful semifinite weight w on $\mathcal{A}$ (see [KaR], II.13). Let (θ_s) be the dual automorphism group on $\mathcal{M}$. It is well known that $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{M}$ and that the position of $\mathcal{A}$ in $\mathcal{M}$ is determined by the group

(θ_s) in the following sense:

$$\forall x \in \mathcal{M}, \quad x \in \mathcal{A} \iff (\forall s \in \mathbb{R},\ \theta_s(x) = x)$$

Moreover $\mathcal{M}$ is semi-finite and can be canonically equipped with a normal faithful semifinite trace τ such that

$$\forall x \in \mathcal{M}, \quad \tau \circ \theta_s = e^{-s} \tau$$

Note that the von Neumann algebra $\mathcal{M}$ is independent from the choice of the n. s. f. weight w on $\mathcal{A}$ (up to a $*$-isomorphism preserving the trace and the group (θ_s)).

Let $L_0(\mathcal{M}, \tau)$ be the space of measurable operators associated with τ (in Nelson's sense [N]). Recall that $L_0(\mathcal{M}, \tau)$ is the completion of $\mathcal{M}$, when $\mathcal{M}$ is equipped with the vector space topology given by the neighborhoods of the origin:

$$N(\varepsilon, \delta) = \{x \in \mathcal{M} \mid \exists e \in \mathcal{M} \text{ projection s. t. } \|xe\| \leq \varepsilon \text{ and } \tau(e^{\perp}) < \delta\}$$

Then the operations on $\mathcal{M}$ extend by continuity to $L_0(\mathcal{M}, \tau)$, which becomes a topological *-algebra.

Note that if $\mathcal{M}$ acts on a Hilbert space H, $L_0(\mathcal{M}, \tau)$ can be identified with a class of unbounded, closed, densely defined operators on H affiliated with $\mathcal{M}$. The operations on $L_0(\mathcal{M}, \tau)$ are identified with the strong sum and the strong product of unbounded operators (i. e. the sum, resp. the product followed by the closure operation).

If h is an element of $L_0(\mathcal{M}, \tau)$, we define its *left support* $\ell(h)$ (resp *right support* $r(h)$) as the least projection e of $\mathcal{M}$ such that $eh = h$ (resp. $he = h$). Clearly $\ell(h^*) = r(h)$, so if h is self-adjoint, $\ell(h) = r(h)$ which we call then simply the *support* of h and denote by $s(h)$.

The space $L_0(\mathcal{M}, \tau)$ is equipped with a positive cone

$$L_0(\mathcal{M}, \tau)_+ = \{h^* h \mid h \in L_0(\mathcal{M}, \tau)\}$$

which is the completion of the positive cone of $\mathcal{M}$. Every element $h \in L_0(\mathcal{M}, \tau)$ has a unique polar decomposition

$$h = u\,|h|$$

where $|h| = (h^* h)^{1/2} \in L_0(\mathcal{M}, \tau)_+$ and u is a partial isometry of $\mathcal{M}$ whose right support is equal to that of h.

The *-automorphisms θ_s, $s \in \mathbb{R}$ extend to *-automorphisms of $L_0(\mathcal{M}, \tau)$. For $0 < p \leq \infty$, the space $L_p(\mathcal{A})$ is defined by

$$L_p(\mathcal{A}) = \{h \in L_0(\mathcal{M}, \tau) \mid \theta_s(h) = e^{-s/p} h\}$$

The space $L_\infty(\mathcal{A})$ coincides with $\mathcal{A}$ (modulo the inclusions $\mathcal{A} \subset \mathcal{M} \subset L_0(\mathcal{M}, \tau)$). The spaces $L_p(\mathcal{A})$ are closed self-adjoint linear subspaces of $L_0(\mathcal{M}, \tau)$. They are closed under left and right multiplications by elements of $\mathcal{A}$. If $h = u\,|h|$ is the polar decomposition of $h \in L_0(\mathcal{M}, \tau)$, then

$$h \in L_p(\mathcal{A}) \iff u \in \mathcal{A} \text{ and } |h| \in L_p(\mathcal{A})$$

As a consequence, the left and right supports of $h \in L_p(\mathcal{A})$ belong to $\mathcal{A}$.

It was shown by Haagerup that there is a linear homeomorphism $\varphi \mapsto h_\varphi$ from $\mathcal{A}_*$ onto $L_1(\mathcal{A})$ (equipped with the vector space topology inherited from $L_0(\mathcal{M}, \tau)$),

and this homeomorphism preserves the additional structure (conjugation, positivity, polar decomposition, action of $\mathcal{A}$). It permits to transfer the norm of $\mathcal{A}_*$ to a norm on $L_1(\mathcal{A})$, denoted by $\| \ \|_1$.

The space $L_1(\mathcal{A})$ is equipped with a distinguished bounded positive linear form Tr, the "trace", defined by

$$\forall \varphi \in \mathcal{A}_*, \quad \mathrm{Tr}\,(h_\varphi) = \varphi(\mathbf{1})$$

Consequently, $\|h\|_1 = \mathrm{Tr}\,(|h|)$ for every $h \in L_1(\mathcal{A})$.

For every $0 < p < \infty$, the Mazur map $\mathcal{A}_+ \to \mathcal{A}_+$, $x \mapsto x^p$ extends by continuity to a map $L_0(\mathcal{M},\tau)_+ \to L_0(\mathcal{M},\tau)_+$, $h \mapsto h^p$. Then

$$\forall h \in L_0(\mathcal{M},\tau)_+ \quad h \in L_p(\mathcal{A}) \Longleftrightarrow h^p \in L_1(\mathcal{A})$$

For $h \in L_p(\mathcal{A})$ set $\|h\|_p = \| \, |h|^p \|_1^{1/p}$. Then $\| \ \|_p$ is a norm when $1 \le p < \infty$, and a p-norm when $0 < p < 1$ (see [Ko3] for this case). The associated vector space topology coincides with that inherited from $L_0(\mathcal{M},\tau)$.

Another important link between the spaces $L_p(\mathcal{A})$ is the *external product*: in fact the product of $L_0(\mathcal{M},\tau)$, $(h,k) \mapsto h \cdot k$, restricts to a bounded bilinear map $L_p(\mathcal{A}) \times L_q(\mathcal{A}) \to L_r(\mathcal{A})$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. This bilinear map has norm one ("non commutative Hölder inequality").

Assume that $\frac{1}{p} + \frac{1}{q} = 1$. Then the bilinear form $L_p(\mathcal{A}) \times L_q(\mathcal{A}) \to \mathbb{C}$, $(h,k) \mapsto \mathrm{Tr}\,(h \cdot k)$ defines a duality bracket between $L_p(\mathcal{A})$ and $L_q(\mathcal{A})$, for which $L_q(\mathcal{A})$ is (isometrically) the dual of $L_p(\mathcal{A})$ (if $p \neq \infty$); moreover we have the tracial property:

$$\forall h \in L_p(\mathcal{A}), k \in L_q(\mathcal{A}), \quad \mathrm{Tr}\,(hk) = \mathrm{Tr}\,(kh)$$

Definition 1.1. *i) Two elements $h, k \in L_0(\mathcal{M},\tau)$ are called* disjoint*, written as $h \perp k$, if they have disjoint left, resp. right supports:*

$$\ell(h) \perp \ell(k) \quad \text{and} \quad r(h) \perp r(k)$$

ii) A sequence $(h_n) \subset L_0(\mathcal{M},\tau)$ is called disjoint *if the h_n's are pairwise disjoint; if in addition $(h_n) \subset L_p(\mathcal{A})$ $(0 < p < \infty)$, (h_n) is called* almost disjoint *if there is a disjoint sequence (h'_n) such that $\lim_n \|h_n - h'_n\|_p = 0$.*

Note that if (h_n) is almost disjoint in $L_p(\mathcal{A})$, so is $(h_{\pi(n)})$ for every permutation π on $\mathbb{N}$; thus we can speak of an almost disjoint countable subset in $L_p(\mathcal{A})$.

We shall repeatedly use of the following two facts.

Fact 1.2. *i) If $h \in L_0(\mathcal{M},\tau)_+$ and $0 < p < \infty$, then $s(h^p) = s(h)$.*
ii) If $h, k \in L_0(\mathcal{M},\tau)$, then $hk = 0$ iff $r(h) \perp \ell(k)$.

Proof: This is easy via a realization of $L_0(\mathcal{M},\tau)$ as a set of of unbounded, closed, densely defined operators on a Hilbert space H. Then $\ell(h)$, resp. $r(h)^\perp$ is the projection onto the closure of the range of h, resp. onto the kernel of h. If $h \in L_0(\mathcal{M},\tau)_+$, then it is self-adjoint and property i) is well known. Concerning ii) we note that if $hk = 0$, then $\mathrm{ran}\,(k) \subset \ker h$, so $\ell(k) \le r(h)^\perp$; conversely if $r(h) \perp \ell(k)$ then $hk = hr(h)\ell(k)k = 0$. $\square$

Fact 1.3. *Let $0 < p < \infty$ and h, k be two elements of $L_p(\mathcal{A})$.*
i) If $h \perp k$, then $\|h+k\|_p^p = \|h\|_p^p + \|k\|_p^p$.
ii) Conversely if $p \neq 2$ and $\|h+k\|_p^p = \|h-k\|_p^p = \|h\|_p^p + \|k\|_p^p$, then $h \perp k$.

Proof: i) If $h \perp k$, then $|h+k|^p = |h|^p + |k|^p$, hence $\mathrm{Tr}\,|h+k|^p = \mathrm{Tr}\,|h|^p + \mathrm{Tr}\,|k|^p$.

ii) These two equalities implies that h, k verify the equality case of Clarkson's inequality. By Theorem A1 of the Appendix, these elements are disjoint. $\square$

We finally mention the following "localization" fact which will be used in section 2.

Fact 1.4. *If e is an arbitrary projection of $\mathcal{A}$, then the subspace $eL_p(\mathcal{A})e$ is isometrically isomorphic to $L_p(e\mathcal{A}e)$, the L_p-space associated with the reduced von Neumann algebra $e\mathcal{A}e$; this isomorphism preserves the bimodule structure (over $e\mathcal{A}e$) as well as the external product in the L_p scale; in particular, it preserves the disjointness.*

This is easily seen by taking a special n. s. f. weight of the form

$$w(x) = w_1(exe) + w_2(e^{\perp}xe^{\perp})$$

where w_1, w_2 are n. s. f. weights on $e\mathcal{A}e$, resp. $e^{\perp}\mathcal{A}e^{\perp}$. Then it is easy to see that e is invariant under the automorphism group (σ_t^w) associated with w and that $\sigma_t^{w_1}$ is nothing but the restriction of σ_t^w to $e\mathcal{A}e$. Thus the crossed product $\mathcal{M}_e$ associated with $e\mathcal{A}e$ is nothing but $e\mathcal{M}e$, on which the dual automorphism group (θ_s^e) is simply the restriction of (θ_s) and the trace τ_e is the restriction of τ.

Ultrapowers of non-commutative L_p-spaces

Let $\mathcal{U}$ be an ultrafilter over some index set I. If X is a Banach space, let $\ell_\infty(I;X)$ be the Banach space of bounded families of elements of X, indexed by I, equipped with the usual supremum norm. Let $N^{\mathcal{U}}$ be the subspace of $\mathcal{U}$-vanishing families, i.e.

$$N^{\mathcal{U}} = \{(x_i)_{i\in I} \in \ell_\infty(I;X) \mid \lim_{i,\mathcal{U}} \|x_i\| = 0\}$$

The ultrapower $X_{\mathcal{U}}$ is simply the quotient Banach space $\ell_\infty(I;X)/N^{\mathcal{U}}$. If $(x_i)_{i\in I}$ is a member of $\ell_\infty(I;X)$, we denote by $(x_i)^\bullet$ its image by the canonical surjection $\ell_\infty(I;X) \to X_{\mathcal{U}}$. The norm of this later element is simply given by $\|(x_i)^\bullet\| = \lim_{i,\mathcal{U}} \|x_i\|$.

The space X is canonically isometrically embedded into its ultrapower $X_{\mathcal{U}}$ by the diagonal embedding $x \mapsto (x)_i^\bullet$, where $(x)_i$ is the constant family (all the members of which are equal to x). We set $\hat{x} = (x)_i^\bullet$. Sometimes we omit the hat over x when no confusion can occur. If X is not finite dimensional and the ultrafilter is not trivial (i.e. not principal) then $X \neq X_{\mathcal{U}}$.

If X, Y are Banach spaces and $T : X \to Y$ is a bounded linear operator, we can define canonically the ultrapower $T_{\mathcal{U}}$ of T as the operator $X_{\mathcal{U}} \to Y_{\mathcal{U}}$, $(x_i)^\bullet \mapsto (Tx_i)^\bullet$. More generally we can define analogously the ultrapower $F_{\mathcal{U}}$ of a locally uniformly continuous map $F : X \to Y$. In particular, if $B : X \times Y \to Z$ is a bounded bilinear map, it has an ultrapower map $B_{\mathcal{U}} : X_{\mathcal{U}} \times Y_{\mathcal{U}} \to Z_{\mathcal{U}}$ defined by $B(\xi, \eta) = (B(x_i, y_i))^\bullet$ whenever $\xi = (x_i)^\bullet, \eta = (y_i)^\bullet$.

All these are also valid for quasi-Banach spaces.

Now let $\mathcal{A}$ be a C*-algebra. Then $\mathcal{A}_{\mathcal{U}}$ is an involutive complex Banach algebra when equipped with the natural product and conjugation operations which are the respective ultrapowers of the product and the conjugation operations of $\mathcal{A}$. In fact, $\mathcal{A}_{\mathcal{U}}$ is a C*-algebra since it verifies the axiom $\|xx^*\| = \|x\|^2$ for every $x \in \mathcal{A}_{\mathcal{U}}$, which characterizes C*-algebras among involutive complex Banach algebras. On the other hand, the class of von Neumann algebras (dual C*-algebras) is not closed under ultrapowers. However it was shown by U. Groh [G] that the class of the preduals of von Neumann algebras is closed by ultrapowers.

So if $\mathcal{a}$ is a von Neumann algebra, and $\mathcal{a}_*$ is its (unique) predual, then $(\mathcal{a}_*)_{\mathcal{U}}$ is isometric to the predual of a von Neumann algebra $\mathcal{A}$; moreover, $\mathcal{a}_{\mathcal{U}}$ identifies naturally to a w*-dense subspace of $\mathcal{A}$. In fact, one can require that $\mathcal{a}_{\mathcal{U}}$ be a *-subalgebra of $\mathcal{A}$; then $\mathcal{A}$ is uniquely defined as C*-algebra. Note that the class of the preduals of semi-finite von Neumann algebras is *not* closed under ultrapowers (see [Ra]), which justifies the use of Haagerup L_p-spaces in the present paper.

Groh's theorem was extended by the first author to the class of non-commutative L_p-spaces, for arbitrary positive real p. It was shown in [Ra] that $L_p(\mathcal{a})_{\mathcal{U}}$ is isometrically isomorphic to $L_p(\mathcal{A})$, where $\mathcal{A}$ does not depend on p (it is precisely the dual of $(\mathcal{a}_*)_{\mathcal{U}}$). In fact, the isomorphisms $\Lambda_p : L_p(\mathcal{a})_{\mathcal{U}} \to L_p(\mathcal{A})$ constructed in [Ra] preserve some more structures. Note that on $L_p(\mathcal{a})_{\mathcal{U}}$ there are conjugation, absolute value map, left and right actions of $\mathcal{a}_{\mathcal{U}}$, and external products with other spaces $L_q(\mathcal{a})_{\mathcal{U}}$, which are simply the ultrapowers of the corresponding operations involving respectively $L_p(\mathcal{a})$, $\mathcal{a}$ and $L_q(\mathcal{a})$. Then the identification maps Λ_p preserve:

- conjugation: $\Lambda_p(\tilde{h}^*) = \Lambda_p(\tilde{h})^*$
- absolute values: $\Lambda_p(|\tilde{h}|) = |\Lambda_p(\tilde{h})|$
- $\mathcal{a}_{\mathcal{U}}$-bimodule structure: $\Lambda_p(\tilde{x} \cdot \tilde{h} \cdot \tilde{y}) = \tilde{x} \cdot \Lambda_p(\tilde{h}) \cdot \tilde{y}$
- external product: $\Lambda_r(\tilde{h} \cdot \tilde{k}) = \Lambda_p(\tilde{h}) \cdot \Lambda_p(\tilde{k})$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

for all $\tilde{h} \in L_p(\mathcal{a})_{\mathcal{U}}$, $\tilde{k} \in L_q(\mathcal{a})_{\mathcal{U}}$, $\tilde{x}, \tilde{y} \in \mathcal{a}_{\mathcal{U}}$. We shall frequently use these properties without any further reference.

ℓ_p-sequences in Banach spaces

A basic sequence (x_n) of a Banach space X is *K-equivalent* to the unit ℓ_p-basis iff there are positive reals a, b with $b/a \leq K$ such that

$$a\,(\sum_n |\lambda_n|^p)^{1/p} \leq \|\sum_n \lambda_n x_n\| \leq b\,(\sum_n |\lambda_n|^p)^{1/p}$$

for every system (λ_n) of finitely nonzero complex numbers.

It is *almost 1-equivalent* to the ℓ_p-basis if for some sequence (ε_n) of positive reals such that $\lim_{n\to\infty} \varepsilon_n = 0$, the tail $(x_m)_{m\geq n}$ is $(1+\varepsilon_n)$-equivalent to the ℓ_p-basis.

It is *asymptotically 1-equivalent* to the ℓ_p-basis if for some sequence (ε_n) of positive reals such that $\lim_{n\to\infty} \varepsilon_n = 0$ we have:

$$(\sum_n (1-\varepsilon_n)^p\,|\lambda_n|^p)^{1/p} \leq \|\sum_n \lambda_n x_n\| \leq (\sum_n (1+\varepsilon_n)^p\,|\lambda_n|^p)^{1/p}$$

for every system (λ_n) of finitely nonzero complex numbers. The space spanned by such a sequence is called an asymptotically isometric copy of ℓ_p in the terminology of [DJLT]. Note that the subspace spanned by a sequence which is almost 1-equivalent to the ℓ^p does not contain necessarily an asymptotically isometric copy of ℓ_p (see [DJLT]).

2. Elements in $L_p(\mathcal{a})_{\mathcal{U}}$ which are disjoint from $L_p(\mathcal{a})$

In this section we develop the main tools of the paper (Theorem 2.3, Lemma 2.6 and Theorem 2.7). We also give several characterizations of bounded sequences in $L_p(\mathcal{a})$ which have almost disjoint subsequences.

Pairs of disjoint elements in ultrapowers

Let $\mathcal{a}$ be a von Neumann algebra, $\mathcal{a}_*$ its predual and $(\mathcal{a}_*)_{\mathcal{U}}$ the ultrapower of $\mathcal{a}_*$ relative to an ultrafilter over some index set I. Let $\mathcal{A}$ be the dual von Neumann algebra of $(\mathcal{a}_*)_{\mathcal{U}}$. In this section we shall prove that disjoint elements of $\mathcal{A}_*$, when considered in $(\mathcal{a}_*)_{\mathcal{U}}$, admit pairwise disjoint families of representatives in $\ell_\infty(I;\mathcal{a}_*)$. This property is easy to prove in the commutative case, using the lattice operations in L_1-spaces. The proof we give in the noncommutative case is based on the fact that elements of the algebra $\mathcal{A}$ can be "locally" identified with elements of the ultrapower $\mathcal{a}_{\mathcal{U}}$.

Recall that a projection p in a von Neumann algebra $\mathcal{M}$ is *σ-finite* if it is the support of a normal state. Equivalently, there is $h \in L_2(\mathcal{M})_+$ such that $s(h) = p$.

Proposition 2.1. *For every $x \in \mathcal{A}$ and every σ-finite projection p of $\mathcal{A}$ there is a family $(x_i) \subset \mathcal{a}$ with $\|x_i\| \le \|x\|$ for every $i \in I$ and representing an element $\tilde{x} = (x_i)^\bullet$ of $\mathcal{a}_{\mathcal{U}}$ such that $\tilde{x}p = xp$.*

Proof: Before to start the proof, recall that the positive cone in $L_1(\mathcal{a})_{\mathcal{U}}$ consists of elements representable by a bounded family of nonnegative elements of $L_1(\mathcal{a})$.
Let now $\tilde{k} \in L_2(\mathcal{A}) = L_2(\mathcal{a})_{\mathcal{U}}$, $\tilde{k} \ge 0$, with support p, and set $\tilde{h} = x\tilde{k}$. Note that

$$0 \le \tilde{h}^*\tilde{h} = \tilde{k}x^*x\tilde{k} \le \|x\|^2\,\tilde{k}^2$$

Let $(h_i)_{i\in I}$ be a bounded family in $L_2(\mathcal{a})$ representing $\tilde{h}$. We can find a bounded family $(\ell_i)_{i\in I}$ in $L_1(\mathcal{a})_+$, representing $\tilde{k}^2$ and such that

$$\forall i \in I, \quad h_i^*h_i \le \|x\|^2\,\ell_i$$

For, let $(a_i)_{i\in I}$ be a bounded family in $L_1(\mathcal{a})$ representing $\tilde{k}^2 - \dfrac{\tilde{h}^*\tilde{h}}{\|x\|^2}$: since this later element is positive, we can choose $a_i \ge 0$ for every $i \in I$; then set $\ell_i = a_i + \dfrac{h_i^*h_i}{\|x\|^2}$.
Then for every $i \in I$ there exists $x_i \in \mathcal{a}$ such that

$$h_i = x_i\ell_i^{1/2} \text{ and } \|x_i\| \le \|x\|$$

The bounded family $(\ell_i^{1/2})_{i\in I}$ represents $(\tilde{k}^2)^{1/2} = \tilde{k}$, and so $x\tilde{k} = \tilde{h} = (x_i\ell_i^{1/2})^\bullet = \tilde{x}\tilde{k}$, which implies $\tilde{x}p = xp$. □

Corollary 2.2. *For every $x \in \mathcal{A}$, $x \ge 0$ and every σ-finite projection p of $\mathcal{A}$ there exists a family $(x_i)_{i\in I} \subset \mathcal{a}$ with $0 \le x_i \le \|x\|$ representing an element $\tilde{x}$ of $\mathcal{a}_{\mathcal{U}}$ such that $p\tilde{x}p = pxp$.*

Proof: Applying Proposition 2.1 to $y = x^{1/2}$ and p, we obtain $(y_i) \subset \mathcal{a}$, with $\|y_i\| \le \|y\| = \|x\|^{1/2}$ and $\tilde{y} = (y_i)^\bullet$ satisfying $\tilde{y}p = yp$. Set $x_i = y_i^*y_i$, then $p\tilde{x}p = p\tilde{y}^*\tilde{y}p = py^2p = pxp$. □

The next result states that two disjoint σ-finite projections of $\mathcal{A}$ can be separated by a projection of $\mathcal{a}_{\mathcal{U}}$. It is the key technical result of the paper.

Theorem 2.3. *Let p, q be two disjoint σ-finite projections in $\mathcal{A}$. There exists a family of projections $(r_i)_{i\in I}$ in $\mathcal{a}$ representing a projection $\tilde{r}$ of $\mathcal{a}_{\mathcal{U}}$ such that:*

$$\tilde{r} \geq p \quad \textit{and} \quad \tilde{r}^{\perp} \geq q$$

Proof: Applying the preceding corollary to $x = p$ and the σ-finite projection $s = p + q$, we find $(x_i)_{i \in I} \subset \mathcal{a}$ with $0 \leq x_i \leq \mathbf{1}$ such that $\tilde{x} = (x_i)^\bullet$ verifies:

$$s\tilde{x}s = sps = p$$

Then

$$s(\mathbf{1} - \tilde{x})s = s - p = q$$

Hence:

$$\begin{cases} \tilde{x}s = p + s^{\perp}\tilde{x}s \\ (\mathbf{1} - \tilde{x})s = q + s^{\perp}(\mathbf{1} - \tilde{x})s \end{cases}$$

whence:

$$\begin{cases} \tilde{x}p = p + s^{\perp}\tilde{x}p \\ (\mathbf{1} - \tilde{x})q = q + s^{\perp}(\mathbf{1} - \tilde{x})q \end{cases}$$

Let $\tilde{k} \in L_2(\mathcal{A}) = L_2(\mathcal{a})_{\mathcal{U}}$ with support p. Since $\|\tilde{x}\| \leq 1$, we have

$$\begin{aligned} \|\tilde{k}\|_2^2 \geq \|\tilde{x}s\tilde{k}\|_2^2 &= \|p\tilde{k}\|_2^2 + \|s^{\perp}\tilde{x}p\tilde{k}\|_2^2 \\ &= \|\tilde{k}\|_2^2 + \|s^{\perp}\tilde{x}\tilde{k}\|_2^2 \end{aligned}$$

Therefore, $s^{\perp}\tilde{x}\tilde{k} = 0$, and so $s^{\perp}\tilde{x}p = 0$. Similarly, since $\|\mathbf{1} - \tilde{x}\| \leq 1$, we have $s^{\perp}(\mathbf{1} - \tilde{x})q = 0$. Thus:

$$\tilde{x}p = p \;\text{ and }\; (\mathbf{1} - \tilde{x})q = q$$

For every $i \in I$ consider the spectral projection $r_i = \chi_{[\frac{1}{2},1]}(x_i)$ of x_i associated with the indicator function of the interval $[\frac{1}{2}, 1]$. Note that $r_i = f(x_i)x_i$, where the function f is defined by $f(t) = t^{-1}\chi_{[\frac{1}{2},1]}(t)$. We have $\|f(x_i)\| \leq \|f\|_\infty = 2$, so the family $(f(x_i))_{i \in I}$ defines an element of $\mathcal{a}_{\mathcal{U}}$. Then:

$$(r_i)^\bullet q = (f(x_i))^\bullet (x_i)^\bullet q = 0$$

Similarly, since $r_i^{\perp} = g(\mathbf{1} - x_i)(\mathbf{1} - x_i)$ with $g(t) = t^{-1}\chi_{(\frac{1}{2},1]}(t)$, we deduce

$$(r_i^{\perp})^\bullet p = (g(\mathbf{1} - x_i))^\bullet (\mathbf{1} - x_i)^\bullet p = 0$$

Therefore $\tilde{r} = (r_i)^\bullet$ is a desired projection of $\mathcal{a}_{\mathcal{U}}$. $\square$

Corollary 2.4. *Let $0 < p < \infty$. Two elements $\tilde{h}, \tilde{k}$ of $L_p(\mathcal{A}) = L_p(\mathcal{a})_{\mathcal{U}}$ are disjoint if and only if they admit representative families $(h_i)_{i \in I}$, $(k_i)_{i \in I}$ such that for every $i \in I$, h_i is disjoint from k_i*

Proof: The "if" part is evident. To prove the necessity of the condition assume $\tilde{h}, \tilde{k}$ are disjoint. By Theorem 2.3, we can find projections $\tilde{r} = (r_i)^\bullet, \tilde{s} = (s_i)^\bullet$ in $\mathcal{a}_{\mathcal{U}}$ such that

$$\begin{aligned} \tilde{s} &\geq \ell(\tilde{h}), & \tilde{s}^{\perp} &\geq \ell(\tilde{k}) \\ \tilde{r} &\geq r(\tilde{h}), & \tilde{r}^{\perp} &\geq r(\tilde{k}) \end{aligned}$$

Let $(h_i)_{i\in I}$, resp $(k_i)_{i\in I}$ be two representative families of $\tilde{h}$, resp $\tilde{k}$. Let $h'_i = s_i h_i r_i$ and $k'_i = s_i^{\perp} k_i r_i^{\perp}$. Then $(h'_i), (k'_i)$ are two desired representative families. $\square$

Elements of $L_p(\mathcal{A})_{\mathcal{U}}$ disjoint from $L_p(\mathcal{A})$

We say that $\tilde{h} \in L_p(\mathcal{A})_{\mathcal{U}}$ is disjoint from $L_p(\mathcal{A})$ (considered as subspace of $L_p(\mathcal{A})_{\mathcal{U}}$) if $\tilde{h} \perp k$ for every $k \in L_p(\mathcal{A})$. (This is an abuse of notation; we should write $\tilde{h} \perp \hat{k}$, where $\hat{k} = (k)^{\bullet}$ is the canonical image of k in $L_p(\mathcal{A})_{\mathcal{U}}$; note that the left and right supports of $\hat{k}$ do not coincide with the canonical images in $\mathcal{A}_{\mathcal{U}}$ of the supports of k). Equivalently, $\tilde{h}k = k\tilde{h} = 0$ for every $k \in L_p(\mathcal{A})$. Since $k \in L_p(\mathcal{A})_+$ and $k^{\alpha} \in L_{p/\alpha}(\mathcal{A})_+$ have the same support (in $\mathcal{A}$), another equivalent condition is that $k\tilde{h} = 0 = \tilde{h}k$ for every k in $L_q(\mathcal{A})$, for some (every) q, $0 < q < \infty$.

A simple example is given by the following lemma:

Lemma 2.5. *Suppose that $\mathcal{U}$ is a free ultrafilter over $\mathbb{N}$ and let $(h_n)_{n\in\mathbb{N}}$ be a bounded disjoint sequence in $L_p(\mathcal{A})$. Then the element $\tilde{h}$ defined by this sequence in $L_p(\mathcal{A})_{\mathcal{U}}$ is disjoint from $L_p(\mathcal{A})$.*

Proof: The left supports $s_n = \ell(h_n)$ are pairwise disjoint. If $k \in L_2(\mathcal{A})$ we have then $\|ks_n\|_2 \to 0$. For, since the elements ks_n, $n \in \mathbb{N}$ are pairwise orthogonal for the natural scalar product of $L_2(\mathcal{A})$:

$$\sum_n \|ks_n\|_2^2 = \|\sum_n ks_n\|_2^2 \leq \|k\|_2^2$$

Consequently, by the Hölder inequality (with $1/r = 1/2+1/p$), $\|kh_n\|_r \leq \|ks_n\|_2 \|h_n\|_p \to 0$. Similarly, $\|h_n k\|_r \to 0$. A fortiori, $\lim_{n,\mathcal{U}} \|h_n k\|_r = 0 = \lim_{n,\mathcal{U}} \|kh_n\|_r = 0$, which implies $k\tilde{h} = \tilde{h}k = 0$. $\square$

Lemma 2.6. *Let $0 < p < \infty$, and let $\mathcal{S}$ be a separable subset of elements of $L_p(\mathcal{A})_{\mathcal{U}}$ which are disjoint from $L_p(\mathcal{A})$. For each $\tilde{h} \in \mathcal{S}$ let $(h_i)_{i\in I}$ be a bounded family in $L_p(\mathcal{A})$ defining $\tilde{h}$. Then for every finite system $\mathcal{P}$ of pairwise commuting projections of $\mathcal{A}$ and every separable subset $\mathcal{K}$ of $L_p(\mathcal{A})$ there exists a family (s_i) of projections of $\mathcal{A}$ commuting with $\mathcal{P}$ and such that:*

$$\begin{cases} \forall k \in \mathcal{K}, & \|s_i k\|_p + \|ks_i\|_p \xrightarrow[i,\mathcal{U}]{} 0 \\ \forall \tilde{h} \in \mathcal{S}, & \left\|s_i^{\perp} h_i\right\|_p + \left\|h_i s_i^{\perp}\right\|_p \xrightarrow[i,\mathcal{U}]{} 0 \end{cases}$$

Proof: Let $\mathcal{P} = \{p_1, ..., p_N\}$: replacing $\mathcal{P}$ by the set of atoms of the (finite) Boolean algebra generated by $\mathcal{P}$, we may suppose that the p_j 's are disjoint and $\sum_{j=1}^{N} p_j = \mathbf{1}$. Note that for every $j = 1, ..., N$, and $\tilde{h} \in \mathcal{S}$ the elements $\hat{p}_j\tilde{h}$ and $\tilde{h}\hat{p}_j$ are disjoint from $L_p(\mathcal{A})$, and a fortiori from $\hat{p}_j L_p(\mathcal{A})\hat{p}_j$. We may identify $p_j L_p(\mathcal{A}) p_j$ with $L_p(p_j \mathcal{A} p_j)$, and $\hat{p}_j L_p(\mathcal{A})_{\mathcal{U}}\hat{p}_j$ with $L_p(p_j\mathcal{A}p_j)_{\mathcal{U}}$. Let

$$e = \bigvee_{\tilde{h}\in\mathcal{S}} \ell(\hat{p}_j\tilde{h}) \vee r(\tilde{h}\hat{p}_j) \qquad f = \bigvee_{k\in\mathcal{K}} \ell(\hat{p}_j\hat{k}) \vee r(\hat{k}\hat{p}_j)$$

Then e and f are σ-finite disjoint projections. Note that all the support projections above are smaller than $\hat{p}_j$, hence belong to $\hat{p}_j\mathcal{A}\hat{p}_j$, and so do e and f. Thus by Theorem 2.3 there

exists a family $(s_i^{(j)})_i$ of projections of $p_j \mathcal{a} p_j$ such that the corresponding projections $\tilde{s}^{(j)}$ of $\hat{p}_j \mathcal{A} \hat{p}_j$ satisfy $e \leq \tilde{s}^{(j)}$ and $f \leq \left(\tilde{s}^{(j)}\right)^{\perp}$.

We set

$$s_i = \sum_{j=1}^{N} s_i^{(j)} = \sum_{j=1}^{N} p_j s_i^{(j)} p_j.$$

Then all s_n clearly commute with $\mathcal{P}$, and

$$(s_i h_i)^{\bullet} = \sum_{j=1}^{N} \tilde{s}^{(j)} \tilde{h} = \sum_{j=1}^{N} \tilde{s}^{(j)} \hat{p}_j \tilde{h} = \sum_{j=1}^{N} \hat{p}_j \tilde{h} = \tilde{h}, \text{ for every } \tilde{h} \in \mathcal{S}$$

$$(s_i k)^{\bullet} = \sum_{j=1}^{N} \tilde{s}^{(j)} \tilde{k} = \sum_{j=1}^{N} \tilde{s}^{(j)} \hat{p}_j \tilde{k} = 0, \text{ for every } k \in \mathcal{K}$$

Similarly, $(h_i s_i)^{\bullet} = \tilde{h}$ and $(k s_i)^{\bullet} = 0$ for all $\tilde{h} \in \mathcal{S}$ and $k \in \mathcal{K}$. Therefore, the family (s_i) satisfies all requirements of the lemma. $\square$

Recall that a sequence (h_n) in $L_p(\mathcal{a})$ is almost disjoint if there is a disjoint sequence $(h'_n) \subset L_p(\mathcal{a})$ such that $\lim_n \|h_n - h'_n\|_p = 0$ (see Definition 1.1).

Theorem 2.7. *A bounded family $(h_i)_{i \in I}$ in $L_p(\mathcal{a})$ has an almost disjoint countable subfamily if and only if for some free ultrafilter $\mathcal{U}$ over I $(h_i)_{i \in I}$ defines an element of the ultrapower $L_p(\mathcal{a})_{\mathcal{U}}$ which is disjoint from $L_p(\mathcal{a})$.*

Proof: The "only if" part results from Lemma 2.5, choosing an ultrafilter $\mathcal{U}$ containing as an element the infinite subset of I indexing the countable subfamily. Let us prove the "if" part.

We use Lemma 2.6 to construct inductively a sequence of distinct indices (i_n) and a sequence (q_n) of commuting projections of $\mathcal{a}$, such that

$$\forall n \in \mathbb{N} : \|q_n^{\perp} h_{i_n}\|_p + \|h_{i_n} q_n^{\perp}\|_p < 2^{-n} \text{ and } \forall m < n, \ \|q_n h_{i_m}\|_p + \|h_{i_m} q_n\|_p < 2^{-n}$$

We start with some $i_0 \in I$ and $q_0 = \mathbf{1}$. At the $(n+1)$-th step apply Lemma 2.6 to $\mathcal{P} = \{q_0, ..., q_n\}$ and $\mathcal{K} = \{h_{i_0}, ..., h_{i_n}\}$.

Set $p_n = q_n(\bigwedge_{k>n} q_k^{\perp})$. The projections p_n are disjoint. Note that since the q_k's commute, we have $\bigvee_{k>n} q_k = \sum_{k>n} x_n q_k$ for some $x_n \in \mathcal{a}$, $0 \leq x_n \leq \mathbf{1}$. Then:

$$\left\|(p_n^{\perp} - q_n^{\perp}) h_{i_n}\right\|_p = \|(q_n - p_n) h_{i_n}\|_p = \|q_n(\bigvee_{k>n} q_k) h_{i_n})\|_p \leq \|(\sum_{k>n} x_n q_k) h_{i_n}\|_p$$

Therefore, if $p \geq 1$,

$$\left\|(p_n^{\perp} - q_n^{\perp}) h_{i_n}\right\|_p \leq \sum_{k>n} \|q_k h_{i_n}\|_p \leq 2^{-n};$$

similarly, if $0 < p < 1$,

$$\left\|(p_n^{\perp} - q_n^{\perp}) h_{i_n}\right\|_p \leq (2^p - 1)^{-1/p} 2^{-n}.$$

Thus in both cases, $\left\|(p_n^{\perp} - q_n^{\perp}) h_{i_n}\right\|_p \underset{n \to \infty}{\longrightarrow} 0$. Hence it follows that $\left\|p_n^{\perp} h_{i_n}\right\|_p \to 0$. In the same way, we show that $\left\|h_{i_n} p_n^{\perp}\right\|_p \to 0$. Therefore, $\|h_{i_n} - p_n h_{i_n} p_n\|_p \to 0$. $\square$

Remarks 2.8: i) Theorem 2.7 has a close analog for left (resp. right) disjointness. Say that a sequence (h_n) in $L_p(\mathcal{A})$ is almost left (resp. right) disjoint if there exists a sequence (h'_n) of pairwise left (resp. right) disjoint vectors in $L_p(\mathcal{A})$ such that $\|h_n - h'_n\|_p \to 0$. Similarly, an element $\tilde{h}$ of the ultrapower $L_p(\mathcal{A})_{\mathcal{U}}$ is left (resp. right) disjoint from $L_p(\mathcal{A})$ if it is left (resp. right) disjoint from every element of $L_p(\mathcal{A})$ (canonically embedded in) $L_p(\mathcal{A})_{\mathcal{U}}$. Then a bounded family in $L_p(\mathcal{A})$ has an almost left (resp. right) disjoint countable subfamily iff for some free ultrafilter $\mathcal{U}$ over the index set I it defines an element of the ultrapower $L_p(\mathcal{A})_{\mathcal{U}}$ which is left (resp. right) disjoint from $L_p(\mathcal{A})$.

ii) The proof of Theorem 2.7 shows in fact the following: if (h_n) is an almost disjoint sequence in $L_p(\mathcal{A})$, then there exist a subsequence (h_{i_n}) and a disjoint sequence (p_n) of projections in $\mathcal{A}$ such that $\|h_{i_n} - p_n h_{i_n} p_n\|_p \to 0$. The same remark also applies to left and right almost disjoint sequences.

Disjoint types over $L_p(\mathcal{A})$

Recall that following [KrM] a *type* over a Banach space E (or p-Banach space) is a function $\tau : E \to \mathbb{R}_+$ of the form $\tau(x) = \lim_{i,\mathcal{U}} \|x + x_i\|$, where (x_i) is a bounded family of points of E and $\mathcal{U}$ an ultrafilter over I. Equivalently, we have $\tau(x) = \|x + \xi\|$, where ξ is an element of the ultrapower $E_{\mathcal{U}}$: then we say that ξ defines the type τ. We say that *a sequence $(x_n) \subset E$ defines the type τ if $\tau(x) = \lim_{n\to\infty} \|x + x_n\|$*. Note that in separable spaces every type is definable by a sequence. We call a type τ over $L_p(\mathcal{A})$ a *disjoint type* if it is definable by an element ξ of some ultrapower $L_p(\mathcal{A})_{\mathcal{U}}$ which is disjoint from $L_p(\mathcal{A})$.

Proposition 2.9. *If $0 < p < \infty$, the disjoint types over $L_p(\mathcal{A})$ are exactly the functions of the form $h \mapsto F_a(h) = (\|h\|^p + a^p)^{1/p}$, where a is a nonnegative real number. Moreover if $p \neq 2$ an element ξ of an ultrapower $L_p(\mathcal{A})_{\mathcal{U}}$ defines a disjoint type over $L_p(\mathcal{A})$ if and only if it is disjoint from $L_p(\mathcal{A})$.*

Proof: If τ is a disjoint type defined by $\xi \in L_p(\mathcal{A})_{\mathcal{U}}$ with $\xi \perp L_p(\mathcal{A})$, we clearly have $\tau = F_a$ with $a = \|\xi\|$. Conversely, if ξ defines the type F_a, then $a = \|\xi\|^p$ and for every $h \in L_p(\mathcal{A})$:

$$\|h \pm \xi\|^p = F_a(\pm h)^p = (\|h\|^p + \|\xi\|^p)$$

hence $\xi \perp h$ by Fact 1.3 when $p \neq 2$. □

Lemma 2.10. *Every disjoint normalized sequence (h_n) in $L_p(\mathcal{A})$ defines a disjoint type.*

Proof: Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. By lemma 2.5, $\tilde{h} = (h_n)^\bullet$ is disjoint from $L_p(\mathcal{A})$. Hence for every $k \in L_p(\mathcal{A})$, $\lim_{n,\mathcal{U}} \|k + h_n\| = \|k + \tilde{h}\| = (\|k\|^p + \|\tilde{h}\|^p)^{1/p} = F_1(k)$. Since this is true for every ultrafilter $\mathcal{U}$, we have $\lim_{n\to\infty} \|k + h_n\| = F_1(k)$. □

The following gives several characterizations of a bounded sequence which defines a disjoint type:

Proposition 2.11. *Let $0 < p, q < \infty$, $p \neq 2$ and (h_n) be a bounded sequence in $L_p(\mathcal{A})$. Assume that the sequence of norms $(\|h_n\|)$ converges. Then the following assertions are equivalent:*
i) (h_n) defines a disjoint type.
ii) For every element h of $L_q(\mathcal{A})$ we have $\lim_{n\to\infty} h \cdot h_n = 0 = \lim_{n\to\infty} h_n \cdot h$.
iii) Every subsequence of (h_n) contains a subsequence which is almost disjoint in $L_p(\mathcal{A})$.

iv) Every subsequence of (h_n) contains a subsequence asymptotically 1-equivalent to the ℓ_p-basis (up to a constant factor).
v) (For $p \geq 1$:) Every subsequence of (h_n) contains a subsequence asymptotically 1-equivalent to the ℓ_p-basis (up to a constant factor) and spanning an almost complemented subspace of $L_p(\mathcal{a})$.

Proof: Note that the hypothesis on the convergence of the norms is necessary since if (h_n) defines a disjoint type F_a, then $\|h_n\| \to F_a(0) = a$.
i) $\Longrightarrow$ ii): For every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the element $\tilde{h}$ defined by (h_n) in $L_p(\mathcal{a})_{\mathcal{U}}$ defines the same disjoint type. By Proposition 2.9, $\tilde{h}$ is disjoint from $L_p(\mathcal{a})$. Equivalently, for every $k \in L_q(\mathcal{a})$ we have $\hat{k}\tilde{h} = 0 = \tilde{h}\hat{k}$, where $\hat{k} = (k)^{\bullet}$ is the canonical image of k in $L_q(\mathcal{a})_{\mathcal{U}}$. Since $\hat{k}\tilde{h} = (kk_n)^{\bullet}$, $\tilde{h}\hat{k} = (h_n k)^{\bullet}$, we have $\lim\limits_{n,\mathcal{U}} kh_n = 0 = \lim\limits_{n,\mathcal{U}} h_n k$. Since this is true for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, we have $\lim\limits_{n\to\infty} kh_n = 0 = \lim\limits_{n\to\infty} h_n k$.
ii) $\Longrightarrow$ iii): Since every subsequence of (h_n) verifies also hypothesis ii), we may argue with the whole sequence. Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and $\tilde{h}$ the element defined by (h_n) in $L_p(\mathcal{a})_{\mathcal{U}}$. Then $\tilde{h}$ is disjoint from $L_p(\mathcal{a})$, and by Theorem 2.7 (h_n) has an almost disjoint subsequence.
iii) $\Longrightarrow$ iv) (resp. and v) if $p \geq 1$): It is clear that every sequence of normalized pairwise disjoint elements of $L_p(\mathcal{a})$ is isometrically equivalent to the ℓ_p-basis (resp. and spanning a 1-complemented subspace if $p \geq 1$). By standard perturbation techniques (see e. g. [LT], prop. 1. a. 9) one deduces that an almost disjoint sequence of elements whose norm converges to 1 has a subsequence which is almost 1-equivalent to the ℓ_p-basis (resp. and spanning an almost 1-complemented subspace). This subsequence (h'_n) in turn has a subsequence which is asymptotically 1-equivalent to the ℓ_p-basis: This is a consequence of the fact that

$$\forall h \in L_p(\mathcal{a}), \quad \lim_{n\to\infty} \|h + h'_n\| = (\|h\|^p + 1)^{1/p}$$

and by a standard Ascoli type argument, this limit is uniform on the unit ball of every finite dimensional subspace V of $L_p(\mathcal{a})$. Hence for every $\delta > 0$ there exists $N = N(V, \delta)$ such that

$$\forall n \geq N, \forall h \in V, \forall \lambda \in \mathbb{C}, \quad (1-\delta)(\|h\|^p + |\lambda|^p) \leq \|h + \lambda h'_n\|^p \leq (1+\delta)(\|h\|^p + |\lambda|^p)$$

Choose a sequence (δ_n) with $0 < \delta_n < 1$ and $\prod\limits_n (1-\delta_n) > 0$, and define inductively $n_0 = 1 < n_1 < .. < n_k < n_{k+1}...$ by applying the preceding to $V_k = \text{span}\,[h_{n_k} \mid 1 \leq \ell \leq k]$ and setting $n_{k+1} = \max\,[n_k + 1, N(V_k, \delta_k))]$.
iv) $\Longrightarrow$ iii): Suppose that h_n itself is asymptotically equivalent to the ℓ_p-basis. Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and for every $m \in \mathbb{N}$ let $\tilde{h}_m$ be the element of $L_p(\mathcal{a})_{\mathcal{U}}$ defined by the sequence $(h_{m+n})_{n\in\mathbb{N}}$. Then the sequence $(\tilde{h}_m)$ is isometrically equivalent to the ℓ_p-basis. Let us make the identification $L_p(\mathcal{a})_{\mathcal{U}} = L_p(\mathcal{A})$. By Fact 1.3 the elements $\tilde{h}_m$ are disjoint in $L_p(\mathcal{A})$. Let ξ be the element of $L_p(\mathcal{A})_{\mathcal{U}}$ defined by the sequence $(\tilde{h}_m)$. It is disjoint from $L_p(\mathcal{A})$, hence a fortiori from $L_p(\mathcal{a})$. On the other hand, iterated ultrapowers are ultrapowers (relative to the product ultrafilter), i.e. $\xi \in L_p(\mathcal{a})_{\mathcal{U}\times\mathcal{U}}$. Recall that the ultrafilter $\mathcal{U}\times\mathcal{U}$ over $\mathbb{N}\times\mathbb{N}$ is defined by

$$A \in \mathcal{U}\times\mathcal{U} \quad \Longleftrightarrow \quad \{n \mid \{m \in \mathbb{N} \mid (n,m) \in A\} \in \mathcal{U}\} \in \mathcal{U}$$

With this identification we have $\xi = (h_{m+n})^{\bullet}_{m,n}$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijective map and $\mathcal{V}$ be the ultrafilter $f(\mathcal{U} \times \mathcal{U}) = \{f(A) \mid A \in \mathcal{U} \times \mathcal{U}\}$. Set $\varphi(f(m,n)) = m+n$. Then the sequence $(h_{\varphi(i)})$ defines in $L_p(\mathcal{A})_{\mathcal{V}}$ an element disjoint from $L_p(\mathcal{A})$, so by Theorem 2.7 it has an almost disjoint subsequence. Since clearly $\varphi(i) \to \infty$ when $i \to \infty$, a further subsequence will be a subsequence of the initial sequence (h_n).
iii)$\Longrightarrow$ i): by Lemma 2.10. $\square$

Remark. In the case $p = 2$, the relations ii) $\Longleftrightarrow$ iii) $\Longrightarrow$ i) $\Longleftrightarrow$ iv) $\Longleftrightarrow$ v) are still true.

3. Embedding of ℓ_p-sums of finite dimensional spaces

We begin this section by recalling some standard notions from Banach space theory. If (F_n) is a (finite or infinite) sequence of Banach (or quasi-Banach) spaces, their p-direct sum $(\bigoplus_n F_n)_p$ is the space of sequences $(x_n) \in \prod_n F_n$ such that $\sum_n \|x_n\|^p_{F_n}$ converges, equipped with the natural norm $\|(x_n)\| = (\sum_n \|x_n\|^p_{F_n})^{1/p}$. As usual, if all the spaces F_n coincide with a given space F, their p-direct sum is denoted by $\ell_p(F)$ for an infinite sequence, $\ell_p^k(F)$ for a finite sequence with k elements.

A Banach (or quasi-Banach) space X contains uniformly a sequence of Banach (or quasi-Banach) spaces (Y_n) if for some constant K the space X contains for every n a subspace X_n which is K-isomorphic to Y_n; then we say that X contains the Y_n's K-uniformly.

We say that a sequence (E_n) of closed subspaces of the space $L_p(\mathcal{A})$ is *almost disjoint* if there exists a sequence (p_n) of pairwise disjoint projections of $\mathcal{A}$ such that $\lim_{n\to\infty} \|T_n\| = 0$, where T_n is the operator $E_n \to L_p(\mathcal{A})$, $x \mapsto (x - p_n x p_n)$.

The following is one of the main results of this paper.

Theorem 3.1. *Let $0 < p < \infty$, $p \neq 2$, $\mathcal{A}$ be a von Neumann algebra and X a closed subspace of $L_p(\mathcal{A})$. Let (F_n) be a sequence of finite dimensional normed (or quasi-normed) spaces.*
i) If X contains K-uniformly the finite p-direct sums $\ell_p^n(F_j)$, $n, j \geq 1$, then X contains a subspace isomorphic to the infinite p-direct sum $(\bigoplus_j F_j)_p$.
ii) More precisely, under the assumption of i) for every $\varepsilon > 0$ there exists an almost disjoint sequence (E_n) of finite dimensional subspaces of X such that for every n, E_n is $(K+\varepsilon)$-isomorphic to F_n.
iii) If in addition $1 \leq p < \infty$ and X contains the $\ell_p^n(F_j)$ $(n, j \geq 1)$ as uniformly complemented subspaces of $L_p(\mathcal{A})$, then the E_n can be found uniformly complemented, and consequently X contains $(\bigoplus_j F_j)_p$ as a complemented subspace of $L_p(\mathcal{A})$.

This result immediately implies Theorem 0.1:

Proof of Theorem 0.1: To deduce Theorem 0.1 from Theorem 3.1 we need only to note that for every $n, m \geq 1$, the space S_p^{nm} contains isometrically $\ell_p^n(S_p^m)$ (as 1-complemented subspace when $p \geq 1$) by just taking the block-diagonal embedding. $\square$

The remainder of this section is devoted to the proof of Theorem 3.1. We first refine the given embeddings of $\ell_p^n(F_j)$ into X. We denote by (e_i) the natural basis of ℓ_p or ℓ_p^n. If F is a space and $x \in F$, then $e_i \otimes x$ denotes the sequence $(0, 0, ..., 0, x, 0...)$, where x is at the i-th place.

Lemma 3.2. *Let X be a (p-)Banach space and F a finite dimensional (quasi-)normed space. Assume that X contains K-uniformly the spaces $\ell_p^n(F)$, $n \geq 1$. Then for every $\varepsilon > 0$ and every $n \geq 1$ there exists a K-isomorphic embedding $T_{n,\varepsilon} : \ell_p^n(F) \hookrightarrow X$ such that for every nonzero $x \in F$ the sequence $\left(\|T_{n,\varepsilon}(e_i \otimes x)\|^{-1} T_{n,\varepsilon}(e_i \otimes x)\right)_{1\leq i\leq n}$ is $(1+\varepsilon)$-equivalent to the unit basis of ℓ_p^n. If in addition $p \geq 1$ and the initial copies of the $\ell_p^n(F)$ are C-complemented, the new ones $T_{n,\varepsilon}(\ell_p(F))$ are KC-complemented.*

Proof: Given $n \geq 1$ let T_n be a K-isomorphic embedding of $\ell_p^n(F)$ into X. We define canonically a K-isomorphic embedding T of $\ell_p(F)$ into some ultrapower $X_{\mathcal{U}}$ of X by extending the T_n to operators $\ell_p(F) \to X$ (simply set $T_n(e_k \otimes x) = 0$ if $k > n$) and then setting $T(e_k \otimes x) = (T_n(e_k \otimes x))^{\bullet}$.

If $p \geq 1$, we use Krivine's Theorem (see [MS], Theorem 12.4 in the real case; [BL], Ch. 6, Cor. 3 for the complex version): every basic sequence (x_n) in a Banach space contains, for some $q \in [1, +\infty]$, every $\varepsilon > 0$ and every $n \geq 1$, a finite sequence of disjoint blocks $(1+\varepsilon)$-equivalent to the ℓ_q^n-basis. Of course, if the sequence (x_n) is itself K-equivalent to the ℓ_p-basis, then $q = p$.

If $0 < p \leq 1$, we use the the following p-normed space version of the well known James distorsion theorem on ℓ_1: if a p-Banach space has a basis equivalent to the ℓ_p-basis, it contains for every $\varepsilon > 0$ a basic sequence which is $(1+\varepsilon)$-equivalent to the ℓ_p-basis, and consists of successive blocks of the initial basis. We refer to [J] or to [LT], Proposition 2e3 for the proof of James' Theorem in the case $p = 1$. The adaptation of this proof to the case $0 < p < 1$ is straightforward.

Fix a non-zero $\xi \in F$. In both cases, for every $n \geq 1$ we can find a sequence $J_1 < J_2 < \ldots < J_n$ of successive disjoint intervals of $\mathbb{N}$ and systems $(\lambda_{k,j})_{k=1,\ldots,n;j\in J_k}$ such that $\sum_{j\in J_k} |\lambda_{k,j}|^p = 1$ for every k, and such that

$$\forall \rho_1, ..., \rho_n \in \mathbb{C}, \quad \|\sum_{k=1}^{n} \rho_k \sum_{j\in J_k} \lambda_{k,j} T(e_j \otimes \xi)\|^p \overset{(1+\varepsilon)}{\sim} \sum_{k=1}^{n} |\rho_k|^p \| \sum_{j\in J_k} \lambda_{k,j} T(e_j \otimes \xi)\|^p$$

where as usual the abbreviation $a \overset{C}{\sim} b$ means $\max(a/b, b/a) \leq \sqrt{C}$. Let $S_\xi^{(n)}$ be the isometry $\ell_n^p \hookrightarrow \ell^p$ defined by

$$S_\xi^{(n)}(e_k) = \sum_{j\in J_k} \lambda_{k,j} e_j$$

and let

$$T_\xi^{(N,n)} = T_N \circ (S_\xi^{(n)} \otimes Id_F) : \ell_p^n(F) \to X$$

Then clearly for N sufficiently large $T_\xi^{(N,n)}$ is a K-isomorphic embedding and

$$\lim_{N,\mathcal{U}} \|T_\xi^{(N,n)}(\sum_{k=1}^{n} \rho_k e_k \otimes \xi)\|_X^p \overset{1+\varepsilon}{\sim} \lim_{N,\mathcal{U}} \sum_{k=1}^{n} |\rho_k|^p \|T_\xi^{(N,n)}(e_k \otimes \xi)\|_X^p$$

for all $\rho_1, ..., \rho_n \in \mathbb{C}$. Using the compacity of the unit ball of ℓ_p^n one easily deduces that for some $N = N(n, \varepsilon)$

$$\|T_\xi^{(N,n)}(\sum_{k=1}^{n} \rho_k e_k \otimes \xi)\|_X^p \overset{1+2\varepsilon}{\sim} \sum_{k=1}^{n} |\rho_k|^p \|T_\xi^{(N,n)}(e_k \otimes \xi)\|_X^p$$

for all $\rho_1, ..., \rho_n \in \mathbb{C}$. Set $T_\xi^{(n)} = T_\xi^{(N(n,\varepsilon),n)}$. Note that

$$\|T_\xi^{(n)}(\sum_k u_k \otimes \xi)\|_X^p \overset{(1+2\varepsilon)^2}{\sim} \sum_k \|T_\xi^{(n)}(u_k \otimes \xi)\|_X^p$$

for every sequence (u_k) of pairwise disjoint elements of ℓ_p^n.
Note moreover that if some $x \in F$ verifies

$$\|T_n(\sum_k u_k \otimes x)\|_X^p \overset{1+\delta}{\sim} \sum_k \|T_n(u_k \otimes x)\|_X^p$$

for every n and every disjoint sequence (u_k) of ℓ_p^n, then we have for every n:

$$\|T_\xi^{(n)}(\sum_k u_k \otimes x)\|_X^p \overset{1+\delta}{\sim} \sum_k \|T_\xi^{(n)}(u_k \otimes x)\|_X^p$$

too (simply because the $S_\xi^N(u_k)$ are pairwise disjoint too). Let us say that the sequence $(T_\xi^{(n)})$ is a (ε, ξ)-refinement of the sequence (T_n).

Let now $\mathcal{E} = \{\xi_1, ..., \xi_d\}$ be an ε-net in the unit sphere S_F of F, i.e. a maximal set of ε-separated points of S_F. We define successively sequences $(T_{\xi_1}^{(n)}), (T_{\xi_1,\xi_2}^{(n)})$, ..., $(T_{\xi_1,\ldots,\xi_d}^{(n)})$ of K-isomorphic embeddings of $\ell_p^n(F)$ into X. The sequence $((T_{\xi_1}^{(n)}))$ is a (ε, ξ_1)-refinement of the sequence (T_n), and for every $j = 2, ..., d$, the sequence $(T_{\xi_1,\ldots,\xi_j}^{(n)})$ is a (ε, ξ_j)-refinement of the sequence $(T_{\xi_1,\ldots,\xi_{(j-1)}}^{(n)})$. The final operators, which we denote by $T_\mathcal{E}^{(n)}$, are still K-isomorphic embeddings and verify

$$\forall \xi \in \mathcal{E}, \forall (\rho_k) \in \ell_p^n, \quad \|T_\mathcal{E}^{(n)}(\sum_k \rho_k e_k \otimes \xi)\|_X^p \overset{(1+2\varepsilon)^2}{\sim} \sum_k |\rho_k|^p \|T_\mathcal{E}^{(n)}(e_k \otimes \xi)\|_X^p$$

If now $x \in S_F$ is arbitrary, let $\xi \in \mathcal{E}$ with $\|x - \xi\| \le \varepsilon$. For every norm one $(\rho_k) \in \ell_p$ we have by triangular inequality in X (in the Banach case):

$$\left| \|\sum_k \rho_k T_\mathcal{E}^{(n)}(e_k \otimes x)\| - \|\sum_k \rho_k T_\mathcal{E}^{(n)}(e_k \otimes \xi)\| \right| \le \|T_\mathcal{E}^{(n)}\| \, \|\sum_k \rho_k e_k \otimes (x - \xi)\|_{\ell_p(F)} \le \varepsilon K$$

and similarly by triangular inequality in ℓ_p^n, and in X:

$$\left| (\sum_k |\rho_k|^p \|T_\mathcal{E}^{(n)}(e_k \otimes x)\|^p)^{1/p} - (\sum_k |\rho_k|^p \|T_\mathcal{E}^{(n)}(e_k \otimes \xi)\|^p)^{1/p} \right| \le \varepsilon K$$

Then we deduce that $(T_\mathcal{E}^{(n)}(e_k \otimes x))_k$ is $f(\varepsilon, K)$-equivalent to the ℓ_p^n-basis, with $f(\varepsilon, K) \to 1$ when $\varepsilon \to 0$ (we find $f(\varepsilon, K) \le \dfrac{(1+2\varepsilon)^4(1+2K\varepsilon)}{1-2K\varepsilon(1+2\varepsilon)^2}$). Similar estimations hold in the p-normed case.

A careful examination of what has been done shows that each $T_\mathcal{E}^{(n)}$ is deduced from some T_N simply by composing on the right with some $S \otimes \mathrm{Id}_F$, where $S : \ell_p^n \hookrightarrow \ell_p^N$ is an isometry. Note that S maps the basis vectors of ℓ_p^n onto disjoint vectors in ℓ_p^N (if $p = 2$ it is not automatic but results from the construction). If $p \ge 1$ the range of such an isometry is always 1-complemented by some norm one projection Q_S (see [LT], prop. 2.a.1; in fact this projection verifies $\|Q_S(|x|)\|_p \le \|x\|_p$ for every $x \in \ell_p^N$). Then $\widetilde{Q}_S = Q_S \otimes \mathrm{Id}_F$ is a norm one projection in $\ell_p^N(F)$. If P_N is a projection from X onto the range of T_N, then $T_N \widetilde{Q}_S T_N^{-1} P_N$ is a projection from X onto the range of $T_\mathcal{E}^{(n)}$ (with norm $\le K \|P_N\|$). $\square$

Lemma 3.3. *For every $j \geq 1$ let $\mathcal{U}_j$ be a free ultrafilter over I and $(X_{i,j})_{i\in I}$ be a family of d_j-dimensional subspaces of $L_p(\mathcal{A})$ such that $(\prod_i X_{i,j})_{\mathcal{U}_j}$, considered as a subspace of $L_p(\mathcal{A})_{\mathcal{U}_j}$, is disjoint from $L_p(\mathcal{A})$. Let (ε_j) be an arbitrary sequence of positive real numbers. Then there exist a sequence $(i_j)_j$ in I and a sequence (p_j) of pairwise disjoint projections of $\mathcal{A}$ such that:*

$$\forall j \geq 1, \quad \sup\{\|h - p_j h p_j\| \mid h \in X_{i_j,j}; \|h\| \leq 1\} \leq \varepsilon_j.$$

Proof: Given j, a finite system $\mathcal{P}$ of pairwise disjoint projections, and a finite dimensional subspace V of $L_p(\mathcal{A})$, we can obtain, using Lemma 2.6, a family $(s_i)_{i\in I}$ of projections of $\mathcal{A}$ which commute with $\mathcal{P}$ and such that:

i) $\forall k \in V, \qquad \|s_i k\| + \|k s_i\| \underset{i,\mathcal{U}_j}{\longrightarrow} 0$

ii) for every bounded family $(h_i) \in \prod_i X_{i,j}$, $\left\|s_i^{\perp} h_i\right\| + \left\|h_i s_i^{\perp}\right\| \underset{i,\mathcal{U}_j}{\longrightarrow} 0.$

To see this we just note that V and $(\prod_i X_{i,j})_{\mathcal{U}_j}$ are separable since they are finite dimensional.

By compacity of the unit balls of V and of $\prod_i X_{i,j}$, the conditions (i), (ii) clearly imply:

i') $\sup\{\|s_i k\| + \|k s_i\| \mid k \in V, \|k\| \leq 1\} \underset{i,\mathcal{U}_j}{\longrightarrow} 0.$

ii') $\sup\{\left\|s_i^{\perp} h\right\| + \left\|h s_i^{\perp}\right\| \mid h \in X_{i,j}, \|h\| \leq 1\} \underset{i,\mathcal{U}_j}{\longrightarrow} 0.$

Let (δ_j) be a sequence of positive real numbers. Now we construct by induction a sequence (i_j) of distinct indices in I and a sequence of pairwise commuting projections (q_j) such that for every $j \geq 1$

$$\forall h \in \sum_{n \leq j-1} X_{i_n,n}, \quad \|q_j h\| + \|h q_j\| < \delta_j \|h\|$$
$$\forall h \in X_{i_j,j}, \quad \left\|q_j^{\perp} h\right\| + \left\|h q_j^{\perp}\right\| < \delta_j \|h\|$$

We shall consider the convex case ($p \geq 1$), the p-normed case ($0 < p < 1$) being treated analogously. Choose some $i_1 \in I$ and set $q_1 = \mathbf{1}$. Assume constructed $i_1, ..., i_j$ and $q_1, ..., q_j$. Set $V = \sum\limits_{n=1}^{j} X_{i_n,n}$, and let (s_i) be a family verifying the conditions (i'), (ii') above with $j+1$ in place of j. Thus for some $i \in T \setminus \{i_1, ..., i_j\}$ we have:

$$\forall h \in \sum_{n \leq j} X_{i_n,n}, \quad \|s_i h\| + \|h s_i\| < \delta_{j+1} \|h\|$$
$$\forall h \in X_{i,j+1}, \quad \left\|s_i^{\perp} h\right\| + \left\|h s_i^{\perp}\right\| < \delta_{j+1} \|h\|$$

Then set $i_{j+1} = i$ and $q_{j+1} = s_i$. Finally, define $p_j = q_j \bigwedge\limits_{k>j} q_k^{\perp}$. It is easy to check that the two sequences (i_j) and (p_j) satisfy the requirements of the lemma if the δ_j are sufficiently small. $\square$

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1: By Lemma 3.2, we find K-embeddings $T_{j,n}$ of $\ell_p^n(F_j)$ into X such that for each nonzero $x \in F_j$ the sequence $\left(\|T_{j,n}(e_i \otimes x)\|^{-1} T_{j,n}(e_i \otimes x)\right)_{1\leq i\leq n}$ is $(1+1/n)$-equivalent to the ℓ_p^n-basis. Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and define $\widetilde{T}_j : \bigcup\limits_n \ell_p^n(F_j) \hookrightarrow X_{\mathcal{U}}$

by $\widetilde{T}_j(e_i\otimes x) = (T_{j,n}(e_i\otimes x))^\bullet$, with the agreement that $T_{j,n}(e_i\otimes x)=0$ if $i>n$. Then $\widetilde{T}_j$ is a K-embedding into $L_p(\mathcal{A}) = L_p(\mathcal{a})_\mathcal{U}$, that we extend by continuity to the whole of $\ell_p(F_j)$. For every nonzero $x\in F_j$, the sequence $\big(\|\widetilde{T}_j(e_i\otimes x)\|^{-1}\widetilde{T}_j(e_i\otimes x)\big)_i$ is 1-equivalent to the ℓ_p-basis, so it defines in $L_p(\mathcal{A})_\mathcal{U}$ an element disjoint from $L_p(\mathcal{A})$, and a fortiori from $L_p(\mathcal{a})$. We can identify $L_p(\mathcal{A})_\mathcal{U} = \big(L_p(\mathcal{a}_\mathcal{U})\big)_\mathcal{U}$ with $L_p(\mathcal{a})_{\mathcal{U}\times\mathcal{U}}$. Set $S_{j,i,n}: F_j\to X, x\mapsto T_{j,n}(e_i\otimes x)$; for $n\ge i$ the operator $S_{j,i,n}$ induces a K-isomorphic embedding of F_j into X. Moreover, if the initial copies of $\ell_p^n(F_j)$ in X are C-complemented in $L_p(\mathcal{a})$, the ranges $S_{j,i,n}(F_j)$, $n\ge i$, are KC-complemented. For every $x\in F_j$, the double sequence $(S_{j,i,n}(x))_{i,n}$ defines an element of $L_p(\mathcal{a})_{\mathcal{U}\times\mathcal{U}}$ disjoint from $L_p(\mathcal{a})$.

Let $\mathcal{V}$ be the trace of the ultrafilter $\mathcal{U}\times\mathcal{U}$ over the set $D=\{(i,n)\mid n\ge i\}$. We apply Lemma 3.3 to the family $(S_{j,i,n}(F_j))_{(i,n)\in D}$ and the ultrafilter $\mathcal{V}$. Let (i_j,n_j) be a sequence in D and (p_j) be a disjoint sequence of projections of $\mathcal{a}$ satisfying the conclusion of that lemma. From now on we write E_j in place of $S_{j,i_j,n_j}(F_j)$: recall that E_j is K-isomorphic to F_j by some isomorphism $T_j: F_j\to E_j$; and that if we denote by R_j the operator $L_p(\mathcal{a})\to L_p(\mathcal{a})$, $h\mapsto p_jhp_j$ we have $\|(Id-R_j)\,|_{E_j}\|<\varepsilon_j$, which proves assertion ii) of the theorem. In the case iii), we have moreover that E_j is KC-complemented in $L_p(\mathcal{a})$ by some projection P_j.

Now we can easily accomplish the proof of the theorem. The assertion i) of the theorem follows by a standard perturbation argument which we sketch here (in the convex case) for further use in Section 6. Let (p_j) and (E_j) be as before. Then for every finite sequence $(y_j)\in\prod_j E_j$ we have

$$\|\sum_j y_j - p_jy_jp_j\| \le \sum_j \varepsilon_j\|y_j\| \le \sum_j \frac{\varepsilon_j}{1-\varepsilon_j}\|p_jy_jp_j\| \le \varepsilon\sup_j\|p_jy_jp_j\|$$

where $\varepsilon = \sum_{j\ge1}\frac{\varepsilon_j}{1-\varepsilon_j}$ is finite and small if the ε_j's are sufficiently small. On the other hand, since the projections p_j are pairwise disjoint,

$$\|\sum_j p_jy_jp_j\| = (\sum_j\|p_jy_jp_j\|^p)^{1/p}$$

Thus it follows that:

$$(1-\varepsilon)(\sum_j\|p_jy_jp_j\|^p)^{1/p} \le \|\sum_j y_j\| \le (1+\varepsilon)(\sum_j\|p_jy_jp_j\|^p)^{1/p}$$

However

$$\|p_jy_jp_j\|\le\|y_j\| \quad\text{ and }\quad \|p_jy_jp_j\|\ge(1-\varepsilon_j)\,\|y_j\|\ge(1-\varepsilon)\,\|y_j\|$$

Hence:

$$(1-\varepsilon)^2(\sum_j\|y_j\|^p)^{1/p} \le \|\sum_j y_j\| \le (1+\varepsilon)(\sum_j\|y_j\|^p)^{1/p}$$

Assume now w.l.o.g. that $\|T_j^{-1}\|\le1$, $\|T_j\|\le K$ for every $j\ge1$. We define

$$T: F=(\bigoplus_{j\ge1}F_j)_p \to E=\sum_{j\ge1}E_j \;\text{ by }\; T((x_j))=\sum_j T_jx_j$$

Then from the preceding inequalities we deduce that $\left\|T^{-1}\right\| \leq (1-\varepsilon)^{-2}$, $\|T\| \leq (1+\varepsilon)K$. This proves assertion i).

In case iii) of the theorem, since

$$\|(Id - P_j R_j) \mid_{E_j} \| = \|(P_j - P_j R_j) \mid_{E_j} \| < KC\varepsilon_j$$

it follows that $W_j = P_j R_j \mid_{E_j}$ is for small ε_j an isomorphism $E_j \to E_j$, with $\left\|W_j^{-1}\right\| \leq (1-KC\varepsilon_j)^{-1}$. Then $Qx = \sum_j W_j^{-1} P_j R_j x$ defines a bounded projection from X onto $\sum_j E_j$. In fact $\|Qx\|_p^p \leq [(1+\varepsilon)(1-\varepsilon)^{-2}]^p \sum\limits_j \left\|W_j^{-1} P_j R_j x\right\|^p \leq M^p \sum\limits_j \|R_j x\|^p = M^p \|x\|^p$ with $M = KC(1+\varepsilon)(1-\varepsilon)^{-2}(1-KC\varepsilon)^{-1}$. $\square$

4. Equiintegrability and the Subsequence Splitting Lemma

In [R3], N. Randrianantoanina introduced the notion of p-equiintegrable susbset of a non-commutative L_p-space. We give here a seemingly more restrictive definition of p-equiintegrable sets; it will appear later that this definition is in fact equivalent to Randrianantoanina's one.

Definition 4.1. *Let $\mathcal{A}$ be a von Neumann algebra and $0 < p < \infty$. A bounded subset K of $L_p(\mathcal{A})$ is called* p-equiintegrable *if* $\sup\limits_{h\in K} \|e_\alpha h e_\alpha\|_p \underset{\alpha}{\longrightarrow} 0$ *for every net (e_α) of projections of $\mathcal{A}$ which w*-converges to* 0.

Remark 4.2. *Finite subsets of $L_p(\mathcal{A})$ are p-equiintegrable.* In fact, given a net of projections (s_α) w*-converging to 0, let A be the set of positive reals p such that $\|hs_\alpha\|_p \underset{\alpha}{\longrightarrow} 0$ for every $h \in L_p(\mathcal{A})$. By Hölder's inequality, one easily sees that $q \in A$ whenever $0 < q < p$ and $p \in A$. Thus A is an interval whose left endpoint is 0. On the other hand, if $p \in A$, then $2p \in A$ for

$$\|hs_\alpha\|_{2p}^2 = \|s_\alpha h^* h s_\alpha\|_p \leq \|(h^*h)s_\alpha\|_p$$

However, it is clear that $2 \in A$ since $\|hs_\alpha\|_2^2 = \langle h^*h, s_\alpha\rangle$ in the identification of $L_1(\mathcal{A})$ with $\mathcal{A}_*$. Therefore, we deduce that $A = (0, \infty)$.

Lemma 4.3. *Every net (p_α) of σ-finite projections of $\mathcal{A}$ which w*-converges to* 0 *contains a sequence which still w*-converges to* 0.

Proof: Construct inductively a sequence (φ_n) in $\mathcal{A}_*^+$ and a sequence (α_n) such that:

i) $\max\{\varphi_m(p_{\alpha_n}) \mid m = 1, ..., n-1\} < n^{-1}$

ii) φ_n has support p_{α_n} and norm 1.

Set $\psi = \sum_{m=1}^\infty 2^{-m}\varphi_m$, then $s(\psi)$ dominates all the p_{α_n}'s, and clearly $\psi(p_{\alpha_n}) \underset{n\to\infty}{\longrightarrow} 0$. Hence (p_{α_n}) w*-converges to zero. $\square$

Proposition 4.4. *A bounded subset K of $L_p(\mathcal{A})$ is p-equiintegrable if and only if for every sequence (e_n) of projections of $\mathcal{A}$ which w*-converges to* 0 *we have* $\sup\limits_{h\in K} \|e_n h e_n\|_p \underset{n}{\longrightarrow} 0$. *In particular, a subset K of $L_p(\mathcal{A})$ is p-equiintegrable if every countable subset of K is.*

Proof: The condition is clearly necessary. Conversely if K is not equiintegrable, there exists a family $(p_i)_{i\in I}$ of projections of $\mathcal{A}$ and an ultrafilter $\mathcal{U}$ over I such that w*-$\lim\limits_{i,\mathcal{U}} p_i = 0$ but $\lim\limits_{i,\mathcal{U}} \sup\limits_{h\in K} \|p_i h p_i\|_p > \delta > 0$. We can clearly suppose that for some family (h_i) of elements of

K we have $\|p_i h_i p_i\| \geq \delta/2$ for every $i \in I$. Let $p'_i = \ell(p_i h_i p_i) \vee r(p_i h_i p_i)$: then $p'_i \leq p_i$ and (p'_i) w*-converges to 0 with respect to $\mathcal{U}$, each p'_i is σ-finite and $p'_i h_i p'_i = p_i h_i p_i$. By Lemma 4.3, there exists a subsequence (p'_{i_n}) which w*-converges to 0. $\square$

Remark 4.5. *If K is p-equiintegrable, then for all bounded nets (x_α), (y_α) of positive elements of $\mathcal{A}$ which w*-converge to 0, we have $\sup_{h \in K} \|x_\alpha h y_\alpha\|_p \underset{\alpha}{\longrightarrow} 0$.*

Proof: Fix $\varepsilon > 0$ and let $e_{\alpha,\varepsilon}$ be the spectral projection $\chi_{[\varepsilon,+\infty)}(x_\alpha + y_\alpha)$. Since $e_{\alpha,\varepsilon} \leq \varepsilon^{-1}(x_\alpha + y_\alpha)$, we have $e_{\alpha,\varepsilon} \xrightarrow[\alpha]{w^*} 0$, hence $\sup_{h \in K} \|e_{\alpha,\varepsilon} h e_{\alpha,\varepsilon}\|_p \underset{\alpha}{\longrightarrow} 0$. Consequently we have $\sup_{h \in K} \|x_\alpha e_{\alpha,\varepsilon} h e_{\alpha,\varepsilon} y_\alpha\|_p \underset{\alpha}{\longrightarrow} 0$. On the other hand, since $0 \leq x_\alpha \leq (x_\alpha + y_\alpha)$, there exist $c_\alpha \in \mathcal{A}$, $0 \leq c_\alpha \leq \mathbf{1}$, such that $x_\alpha = (x_\alpha + y_\alpha)^{1/2} c_\alpha (x_\alpha + y_\alpha)^{1/2}$. Then

$$\left\|x_\alpha e^\perp_{\alpha,\varepsilon}\right\| \leq \left\|(x_\alpha + y_\alpha)^{1/2} c_\alpha\right\| \left\|(x_\alpha + y_\alpha)^{1/2} e^\perp_{\alpha,\varepsilon}\right\| \leq \varepsilon^{1/2} M^{1/2}$$

where M is a bound for the $\|x_\alpha + y_\alpha\|$. Similarly, $\left\|e^\perp_{\alpha,\varepsilon} y_\alpha\right\| \leq \varepsilon^{1/2} M^{1/2}$. So we obtain $\overline{\lim_\alpha} \sup_{h \in K} \|x_\alpha h y_\alpha\|_p \leq 2\varepsilon^{1/2} M^{3/2} M'$, where M' is a bound for the $\|h\|$, $h \in K$. $\square$

Now we characterize the p-equiintegrability of a bounded sequence in $L_p(\mathcal{A})$ in terms of the element it defines in an ultrapower of $L_p(\mathcal{A})$ and the disjointness of this element from $L_p(\mathcal{A})$. To this end we introduce the following notation. Let $\mathcal{U}$ be an ultrafilter over the index set I. Let s_e be the support of $L_p(\mathcal{A})$ in $L_p(\mathcal{A})_\mathcal{U}$ (considered as a non-commutative L_p-space $L_p(\mathcal{A})$). We have thus (since $L_p(\mathcal{A})$ is self-adjoint and generated by its positive cone):

$$\begin{aligned} s_e &= \sup\{\ell(\hat{h}) \vee r(\hat{h}) \mid h \in L_p(\mathcal{A})\} \\ &= \sup\{\ell(\hat{h}) \mid h \in L_p(\mathcal{A})\} = \sup\{r(\hat{h}) \mid h \in L_p(\mathcal{A})\} \\ &= \sup\{s(\hat{h}) \mid h \in L_p(\mathcal{A})_+\} \end{aligned}$$

It is also clear that s_e does not depend on $p \in (0,\infty)$, since $s(\hat{h}) = s((\hat{h})^p) = s(\widehat{h^p})$ for every $h \in L_p(\mathcal{A})^+$. Note also that an element $\tilde{h} \in L_p(\mathcal{A})_\mathcal{U}$ is disjoint from $L_p(\mathcal{A})$ iff $\tilde{h} = s_e^\perp \tilde{h} s_e^\perp$.

If $\mathcal{A}$ is σ-finite, then $s_e = s(\hat{h}_0)$ for every $h_0 \in L_p(\mathcal{A})_+$ with support $s(h_0) = \mathbf{1}$ (when $p = 1$ this means that the associated $\varphi_0 \in \mathcal{A}_*$ is faithful). For, let $h \in L_p(\mathcal{A})_+$; since $\mathcal{A} \cdot h_0$ is dense in $L_p(\mathcal{A})$, there exists for every $\varepsilon > 0$ an $x \in \mathcal{A}$ such that $\|h - xh_0\| < \varepsilon$. Then $\|\hat{h} - \hat{x}\hat{h}_0\| \leq \varepsilon$ and we see that $\hat{h}$ is in the closure of $\mathcal{A}\hat{h}_0$. So $s(\hat{h}) = r(\hat{h}) \leq r(\hat{h}_0) = s(\hat{h}_0)$.

If $\mathcal{A}$ is a finite von Neumann algebra, then s_e is a central projection. For, assume that there is a finite normal faithful trace τ on $\mathcal{A}$. Let $\hat{\tau} = (\tau)^\bullet$ be its canonical image in $(\mathcal{A}_*)_\mathcal{U}$: then $s_e = s(\hat{\tau})$. For any $\tilde{x}, \tilde{y} \in \mathcal{A}_\mathcal{U}$ we clearly have $\hat{\tau}(\tilde{x}\tilde{y}) = \hat{\tau}(\tilde{y}\tilde{x})$. By the w*-density of $\mathcal{A}_\mathcal{U}$ in $\mathcal{A}$, we deduce that $\hat{\tau}$ is tracial, and consequently its support is central. In the general case, we can argue similarly, using a faithful family of normal traces with pairwise disjoint supports.

The main result of this section is the following. Recall that an ultrafilter $\mathcal{U}$ is *countably incomplete* if there exists a sequence $(A_n)_{n\geq 1}$ of members of $\mathcal{U}$ such that $\bigcap_{n \geq 1} A_n = \emptyset$ (so is every non trivial ultrafilter on a countable set).

Theorem 4.6. *Let $\mathcal{U}$ be a countably incomplete ultrafilter over the set I. Let s_e be the support of $L_p(\mathcal{A})$ in $L_p(\mathcal{A}) = L_p(\mathcal{A})_\mathcal{U}$. Then an element $\tilde{h}$ of $L_p(\mathcal{A})_\mathcal{U}$ verifies the equality $s_e^\perp \tilde{h} s_e^\perp = 0$ if and only if it admits a p-equiintegrable representing family $(h_i)_{i \in I}$.*

For the proof of this theorem we shall need the following density lemma (see also [Ju2] and [JX] for similar results).

Lemma 4.7. *Let h_0 be an element of $L_p(\boldsymbol{a})$, $0 < p < \infty$. Then $\boldsymbol{a} \cdot h_0$ is dense in $L_p(\boldsymbol{a}) \cdot r(h_0)$, and $h_0 \cdot \boldsymbol{a}$ is dense in $\ell(h_0) \cdot L_p(\boldsymbol{a})$.*

Proof: Clearly we can assume w.l.o.g. that h_0 is positive and $\|h_0\|_p = 1$. Then $\ell(h_0) = r(h_0) = s(h_0)$. Assume first that $p \geq 1$ and let q be the conjugate index of p. Then the dual space of $L_p(\boldsymbol{a}) \cdot s(h_0)$ is the space $s(h_0) \cdot L_q(\boldsymbol{a})$ (under the duality $\langle h, k\rangle = \mathrm{Tr}\,(hk)$). If $k \in s(h_0) \cdot L_q(\boldsymbol{a})$ belongs to the annihilator of $\boldsymbol{a} \cdot h_0$, then $\mathrm{Tr}(xh_0k) = 0$ for every $x \in \boldsymbol{a}$, which in turn implies that $h_0 \cdot k = 0$ (as element of $L_1(\boldsymbol{a})$), hence $k = s(h_0) \cdot k = 0$. So the linear space $\boldsymbol{a} \cdot h_0$ is dense in $L_p(\boldsymbol{a}) \cdot s(h_0)$. Similarly, $h_0 \cdot \boldsymbol{a}$ is dense in $s(h_0) \cdot L_p(\boldsymbol{a})$. Assume now that $1/2 \leq p < 1$. Every $h \in L_p(\boldsymbol{a}) \cdot s(h_0)$ can be factorized as:

$$h = u\,|x| = u\,|x|^{1/2}\,|x|^{1/2} = (u\,|x|^{1/2}\,s(h_0)) \cdot (|x|^{1/2}\,s(h_0))$$

since the supports of $|h|$ and $|h|^{1/2}$ coincide with the right support of h, hence are included in $s(h_0)$. So $h = k'k''$ with $k', k'' \in L_{2p}(\boldsymbol{a}) \cdot s(h_0)$. Since $2p \geq 1$, there exists by the preceding argument a sequence (y_n) in $\boldsymbol{a}$ such that $y_n \cdot h_0^{1/2} \to k''$ (for the norm of $L_{2p}(\boldsymbol{a})$); and for every n there exists a sequence $(x_m^n)_m$ in $\boldsymbol{a}$ such that $x_m^n \cdot h_0^{1/2} \to k' y_n s(h_0)$ when $m \to \infty$. Then

$$\lim_{n\to\infty} \lim_{m\to\infty} x_m^n h_0 = \lim_{n\to\infty} k' y_n s(h_0) \cdot h_0^{1/2} = k'k'' = h$$

Therefore the conclusion of the lemma is true for $1/2 \leq p < 1$. Iterating this procedure we see that it is true for every $0 < p < 1$. $\square$

Proof of Theorem 4.6: Let (h_i) is a bounded family in $L_p(\boldsymbol{a})$, and let $\tilde{h}$ be the corresponding element of $L_p(\boldsymbol{a})_{\mathcal{U}}$.

a) Suppose that $s_e^{\perp}\tilde{h}s_e^{\perp} \neq 0$. Then there exists a σ-finite projection $q \in \mathcal{A}$ disjoint from s_e such that $q\tilde{h}q \neq 0$. Let $\|q\tilde{h}q\| = \delta > 0$. By Theorem 2.3, for every $\varphi \in \boldsymbol{a}_*^+$ there exists a family (q_i) of projections of $\boldsymbol{a}$ such that $\tilde{q} := (q_i)^\bullet \geq q$ and $\tilde{q}^{\perp} \geq s(\widehat{\varphi})$. The second inequality yields $\lim\limits_{i,\mathcal{U}} \varphi(q_i) = 0$, and the first one implies $\lim\limits_{i,\mathcal{U}} \|q_i h_i q_i\| \geq \|q\tilde{h}q\| = \delta$. Note that we may suppose that each q_i is σ-finite, by replacing if necessary q_i by $q_i' = \ell(q_ih_iq_i) \vee r(q_ih_iq_i)$. So for each $\varepsilon > 0$ we can find $i \in I$ such that $\|q_ih_iq_i\| \geq \delta/2$ and $\varphi(q_i) < \varepsilon$. Now it is easy to construct inductively a sequence (φ_n) in $\boldsymbol{a}_*^+$, a sequence (i_n) in I, and a sequence (s_n) of σ-finite projections of $\boldsymbol{a}$ such that:

i) $\|s_n h_{i_n} s_n\| \geq \delta/2$
ii) $\max\{\varphi_m(s_n) \mid m = 1, ..., n-1\} < 1/n$
iii) φ_n has support s_n and norm 1.

Set $\psi = \sum_{m=1}^{\infty} 2^{-m}\varphi_m$, then $s(\psi)$ dominates all the s_n's, and clearly $\psi(s_n) \underset{n\to\infty}{\longrightarrow} 0$. Hence (s_n) w*-converges to zero, and by (i) the sequence (h_{i_n}) cannot be p-equiintegrable, and a fortiori the family (h_i) is not p-equiintegrable.

b) Conversely, assume that $s_e^{\perp}\tilde{h}s_e^{\perp} = 0$. Then $\tilde{h} = \tilde{h}s_e + s_e(\tilde{h}s_e^{\perp}) \in L_p(\mathcal{A})s_e + s_eL_p(\mathcal{A})$. By Lemma 4.7, $\mathcal{A} \cdot L_p(\boldsymbol{a}) = \mathrm{span}\,\{x\hat{k} \mid x \in \mathcal{A}, k \in L_p(\boldsymbol{a})\}$ is dense in $L_p(\mathcal{A}) \cdot s_e$, and $L_p(\boldsymbol{a}) \cdot \mathcal{A}$ dense in $s_e \cdot L_p(\mathcal{A})$. Note also that by Proposition 2.1, $\mathcal{A} \cdot L_p(\boldsymbol{a}) = \boldsymbol{a}_{\mathcal{U}} \cdot L_p(\boldsymbol{a})$ and $L_p(\boldsymbol{a}) \cdot \mathcal{A} = L_p(\boldsymbol{a}) \cdot \boldsymbol{a}_{\mathcal{U}}$. If $\tilde{h} \in \boldsymbol{a}_{\mathcal{U}} \cdot L_p(\boldsymbol{a})$, it admits a representing family (h_i) of the type

$(x_i h)$, where (x_i) is a bounded family in $\mathcal{A}$ and $h \in L_p(\mathcal{A})$. Let (s_α) be a net of projections which w*-converges to 0. Then by Remark 4.2,

$$\sup_i \|h_i s_\alpha\|_p = \sup_i \|x_i h s_\alpha\|_p \leq (\sup_i \|x_i\|) \|h e_\alpha\|_p \underset{\alpha}{\longrightarrow} 0$$

Thus (h_i) is p-equiintegrable. Similarly, every $\tilde{h} \in L_p(\mathcal{A}) \cdot \mathcal{A}_{\mathcal{U}}$ has a p-equiintegrable representing family. Hence the proof will be complete if we show that the subspace of $L_p(\mathcal{A})_{\mathcal{U}}$ consisting of elements having a p-equiintegrable representing family is closed.

Let $(\tilde{h}^{(n)})_n$ be a sequence in $L_p(\mathcal{A})_{\mathcal{U}}$ which converges to an element $\tilde{h}$ and suppose that each $\tilde{h}^{(n)}$ admits a p-equiintegrable representing family $(h_i^{(n)})_i$. We may suppose that $\|\tilde{h}^{(n)} - \tilde{h}\| < 1/n$. Let (h_i) be a representing family for $\tilde{h}$. Since $\mathcal{U}$ is countably incomplete, we can find a decreasing sequence $U_1 \supset U_2 \supset ... \supset U_n \supset ...$ of members of $\mathcal{U}$ such that $\bigcap_n U_n = \emptyset$ and

$$i \in U_n \Longrightarrow \|h_i^{(n)} - h_i\| \leq \frac{1}{n}$$

Set $h_i' = h_i^{(n)}$ if $i \in U_n \setminus U_{n+1}$ and $h_i' = 0$ if $i \notin U_1$. Then $\|h_i' - h_i\|_p < n^{-1}$ for every $i \in U_n$, which proves that $(h_i')^\bullet = (h_i)^\bullet = \tilde{h}$. Fix $n \geq 1$ and $i \in U_1$. Let $m \geq 1$ such that $i \in U_m \setminus U_{m+1}$. If $m \geq n$ we have

$$\|h_i' - h_i^{(n)}\| = \|h_i^{(m)} - h_i^{(n)}\| \leq \|h_i - h_i^{(m)}\| + \|h_i - h_i^{(n)}\| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{2}{n}$$

Consequently

$$\forall i \in U_1, \inf_{1 \leq j \leq n} \|h_i' - h_i^{(j)}\| \leq \frac{2}{n} \underset{n \to \infty}{\longrightarrow} 0$$

Using the fact that a finite union of p-equiintegrable sets is p-equiintegrable, we easily see that the family $(h_i')_{i \in U_1}$ is p-equiintegrable. Since $\{h_i' \mid i \in I \setminus U_1\} = \{0\}$ is clearly p-equiintegrable too, we are done. $\square$

Remark. In the proof of Theorem 4.6 the hypothesis that the ultrafiter is countably incomplete was not used for the sufficiency of the condition.

Theorem 4.6 permits one to easily recover the following Subsequence Splitting Lemma obtained by N. Randrianantoanina [R3].

Corollary 4.8. *Let $0 < p < \infty$ and $\mathcal{A}$ be a von Neumann algebra. Let (h_n) be a bounded sequence in $L_p(\mathcal{A})$. Then there exists an increasing sequence (n_k) of integers and a sequence (p_k) of pairwise disjoint projections in $\mathcal{A}$ such that the sequence $(h_{n_k} - p_k h_{n_k} p_k)$ is p-equiintegrable. As a consequence, we have the splitting $h_{n_k} = h_k' + h_k''$, where (h_k') is p-equiintegrable and (h_k'') is disjoint.*

Proof: Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$ and $\tilde{h} = (h_n)^\bullet$. Let $\tilde{h}'' = s_e^\perp \tilde{h} s_e^\perp$ and $\tilde{h}' = \tilde{h} - \tilde{h}''$. Since $s_e^\perp \tilde{h}' s_e^\perp = 0$, by Theorem 4.6 the element $\tilde{h}'$ admits a representing sequence (h_n') which is p-equiintegrable. Let $h_n'' = h_n - h_n'$. Then $(h_n'')^\bullet = \tilde{h}''$. Since $\tilde{h}''$ is disjoint from $L_p(\mathcal{A})$, by Theorem 2.7 there is an increasing sequence (n_k) of integers and a disjoint sequence (p_k) of projections such that $\|h_{n_k}'' - p_k h_{n_k}'' p_k\| \longrightarrow 0$ when $k \to \infty$. Since (h_n') is p-equiintegrable, $\|p_k h_{n_k}' p_k\| \to 0$, and thus it follows that

$$h_{n_k} - p_k h_{n_k} p_k = h_{n_k}' - p_k h_{n_k}' p_k + [h_{n_k}'' - p_k h_{n_k}'' p_k]$$

is a perturbation of a p-equiintegrable sequence by a norm vanishing sequence, so is p-equiintegrable too. $\square$

Corollary 4.9. *The following conditions are equivalent for a bounded subset K of $L_p(\mathcal{A})$:*
i) K is p-equiintegrable;
ii) for every disjoint sequence (p_k) of projections of $\mathcal{A}$, $\lim_{k\to\infty} \sup_{h\in K} \|p_k h p_k\| \to 0$;
iii) for every sequence (p_k) of projections of $\mathcal{A}$ which decreases to 0, $\lim_{k\to\infty} \sup_{h\in K} \|p_k h p_k\| \to 0$.
If in addition $\mathcal{A}$ is σ-finite and φ_0 is a normal faithful state on $\mathcal{A}$, i)-iii) are equivalent to the following:
iv) $\lim_{\varepsilon\to 0} \sup\{\|ehe\| \mid h \in K, e \in \mathcal{A}$ projection such that $\varphi_0(e) \le \varepsilon\} = 0$.

Proof: It is clear that condition (i) implies (ii) and (iii). The equivalence between (ii) and (iii) is easy (see also [R3]). To prove that (ii) implies (i), suppose that K is not p-equiintegrable. Then by Proposition 4.4, it contains a sequence (h_n) which is not p-equiintegrable, and by Corollary 4.8, we can suppose that $h_n = h'_n + p_n h_n p_n$, where (h'_n) is p-equiintegrable and (p_n) is a disjoint sequence of projections of $\mathcal{A}$. Then $\|p_n h_n p_n\|$ does not converges to 0, otherwise (h_n) would be p-equiintegrable. The equivalence of (iv) and (i) is due to the fact that a net (e_α) of projections w*-converges to zero iff $\varphi_0(e_\alpha) \to 0$. This in turn follows from the density of $\mathcal{A} \cdot \varphi_0$ in $\mathcal{A}_*$ and the fact that $x \cdot \varphi_0(e_\alpha) = \varphi_0(e_\alpha x) \le \varphi_0(e_\alpha)^{1/2} \varphi_0(x^*x)^{1/2}$ for every $x \in \mathcal{A}$. $\square$

Remark. Randrianantoanina [R3] took the part iii) of Corollary 4.9 as the definition of p-equiintegrability.

Like for the left resp. right disjoint sequences (see Remark 2.8), we shall also consider the corresponding left resp. right p-equiintegrable sets.

Definition 4.10. *We call a bounded set K of $L_p(\mathcal{A})$* left p-equiintegrable, *resp.* right p-equiintegrable *if for every net (e_α) of projections w*-converging to 0 we have $\sup_{h\in K} \|e_\alpha h\| \xrightarrow[\alpha]{} 0$, resp. $\sup_{h\in K} \|h e_\alpha\| \xrightarrow[\alpha]{} 0$. We say that K is* p-biequiintegrable *if it is both left and right p-equiintegrable.*

In this definition we may again w.l.o.g. replace nets by sequences. Note that if K consists of positive elements, then K is p-equiintegrable iff it is p-biequiintegrable since $\|eh\| = \left\|eh^{1/2}h^{1/2}\right\| \le \|ehe\|\,\|h\|$ for every $h \in L_p(\mathcal{A})^+$ and e projection of $\mathcal{A}$. Thus the four notions of p-equiintegrability coincide on subsets of $L_p(\mathcal{A})^+$. Note also that K is left p-equiintegrable iff $K^* = \{h^* \mid h \in K\}$ is right p-equiintegrable; if K is left (resp. right) p-equiintegrable and B is a bounded subset of $\mathcal{A}$, then the set $\{k \cdot x \mid k \in K, x \in B\}$ (resp. $\{x \cdot k \mid k \in K, x \in B\}$) is left (resp. right) p-equiintegrable too; in particular, K is left (resp. right) p-equiintegrable iff $|K^*| := \{|h^*| \mid h \in K\}$ (resp. $|K| := \{|h| \mid h \in K\}$) is p-equiintegrable. Finally, K is p-biequiintegrable iff both $|K|$ and $|K^*|$ are.

Theorem 4.6 can be refined in the following way:

Proposition 4.11. *Let $\mathcal{U}$ be a countably incomplete ultrafilter over the set I. Let s_e be the support of $L_p(\mathcal{A})$ in $L_p(\mathcal{A}) = L_p(\mathcal{A})_{\mathcal{U}}$. Let $\tilde{h} \in L_p(\mathcal{A})_{\mathcal{U}}$. Then:*
i) $\tilde{h} \in L_p(\mathcal{A})s_e$ iff it admits a right p-equiintegrable representing family.
ii) $\tilde{h} \in s_e L_p(\mathcal{A})$ iff it admits a left p-equiintegrable representing family.
iii) $\tilde{h} \in s_e L_p(\mathcal{A}) s_e$ iff it admits a p-biequiintegrable representing family.

Proof: If $\tilde{h} \notin L_p(\mathcal{A})s_e$, $\tilde{h} s_e^\perp \neq 0$. Then we can prove that $\tilde{h}$ has no right p-equiintegrable representing family in a way very similar to the first part of the proof of Theorem 4.6.

For the converse implication we note that $\mathcal{a}_{\mathcal{U}}L_p(\mathcal{a})$ is dense in $L_p(\mathcal{A})s_e$ (Lemma 4.7) and the space of elements $\tilde{h}$ admitting a right p-equiintegrable representing family is closed. The second assertion follows by conjugation. Finally if $\tilde{h}$ admits a p-biequiintegrable representing family, it belongs to $L_p(\mathcal{A})s_e \cap s_e L_p(\mathcal{A}) = s_e L_p(\mathcal{A})s_e$. For the converse implication, we need only to note that by Lemma 4.7 $L_p(\mathcal{a})\mathcal{a}_{\mathcal{U}}L_p(\mathcal{a})$ is dense in $s_e L_p(\mathcal{A})s_e$ and that the space of elements $\tilde{h}$ admitting a p-biequiintegrable representing family is closed. $\square$

The following corollary improves the Subsequence Splitting Lemma.

Corollary 4.12. *Let $0 < p < \infty$ and $\mathcal{a}$ be a von Neumann algebra. Let (h_n) be a bounded sequence in $L_p(\mathcal{a})$. Then there exist an increasing sequence (n_k) of integers and a disjoint sequence (p_k) of projections in $\mathcal{a}$ such that:*
- the sequence $(p_k^\perp h_{n_k} p_k^\perp)$ is p-biequiintegrable,
- the sequence $(p_k^\perp h_{n_k} p_k)$ is left p-equiintegrable,
- the sequence $(p_k h_{n_k} p_k^\perp)$ is right p-equiintegrable.
Consequently, the sequence (h_{n_k}) splits into the sum of four sequences:

$$h_{n_k} = a_k + b_k + c_k + d_k$$

where (a_k) is p-biequiintegrable, (b_k) is left p-equiintegrable and right disjoint, (c_k) is right p-equiintegrable and left disjoint, (d_k) is disjoint.

Proof: Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. By Remark 2.8, for every bounded sequence (h_n) in $L_p(\mathcal{a})$ defining an element $\tilde{h}$ of $L_p(\mathcal{A})s_e^\perp$ there exist a subsequence (h_{n_k}) and a disjoint sequence of projections (p_k) such that $\|h_{n_k} - h_{n_k}p_k\|_p \longrightarrow 0$. Moreover, given a finite set of bounded sequences $(h_n^{(j)})$, $j = 1, ..., N$, each defining an element of $L_p(A)s_e^\perp$, one can find a *common* increasing sequence (n_k) and a *common* disjoint sequence (p_k) of projections such that $\|h_{n_k}^{(j)} - h_{n_k}^{(j)}p_k\|_p \longrightarrow 0$, $j = 1, ..., N$ (compare with Lemma 3.3).

Now fix a bounded sequence (h_n) in $L_p(\mathcal{a})$, and le $\tilde{h}$ be the corresponding element in $L_p(\mathcal{a})_{\mathcal{U}} = L_p(\mathcal{A})$. According to the decomposition

$$\tilde{h} = s_e\tilde{h}s_e + s_e\tilde{h}s_e^\perp + s_e^\perp\tilde{h}s_e + s_e^\perp\tilde{h}s_e^\perp$$

and by Proposition 4.11, we find four bounded sequences (a_n), (d_n), (c_n) and (d_n) such that $h_n = a_n + b_n + c_n + d_n$, (a_n) p-biequiintegrable and $(a_n)^\bullet = s_e\tilde{h}s_e$, (b_n) left p-equiintegrable and $(b_n)^\bullet = s_e\tilde{h}s_e^\perp$, (c_n) right p-equiintegrable and $(c_n)^\bullet = s_e^\perp\tilde{h}s_e$, and finally $(d_n)^\bullet = s_e^\perp\tilde{h}s_e^\perp$.

Applying the preceding remark to the set $\{(b_n), (c_n^*), (d_n), (d_n^*)\}$, we obtain an increasing sequence (n_k) of integers and a disjoint sequence (p_k) of projections such that the four sequences

$$(b_{n_k} - b_{n_k}p_k),\ (c_{n_k} - p_k c_{n_k}),\ (d_{n_k} - d_{n_k}p_k),\ (d_{n_k} - p_k d_{n_k})$$

all converge to 0. Note that $(p_k b_{n_k})$ and $(c_{n_k}p_k)$ converge to zero too since (b_n) and (c_n) are respectively left and right p-equiintegrable and the projections p_k are pairwise disjoint. Therefore, we deduce that the three sequences $(b_{n_k} - p_k^\perp b_{n_k}p_k)$, $(c_{n_k} - p_k c_{n_k} p_k^\perp)$ and $(d_{n_k} - p_k d_{n_k} p_k)$ converge to zero as well. Thus we can decompose h_{n_k} as:

$$h_{n_k} = a'_k + p_k^\perp b_{n_k} p_k + p_k c_{n_k} p_k^\perp + p_k d_k p_k$$

where (a'_k) is a sequence such that $\|a'_k - a_{n_k}\| \underset{k\to\infty}{\longrightarrow} 0$. Consequently, (a'_k) is p-biequiintegrable too. It follows that $(p_k^\perp h_{n_k} p_k^\perp) = (p_k^\perp a'_k p_k^\perp)$ is p-biequiintegrable, $(p_k^\perp h_{n_k} p_k) = (p_k^\perp a'_k p_k +$

$p_k^{\perp}b_{n_k}p_k)$ is left p-equiintegrable and $(p_kh_{n_k}p_k^{\perp}) = (p_ka'_kp_k^{\perp} + p_kc_{n_k}p_k^{\perp})$ is right p-equiintegrable. $\square$

Remark. Using Corollary 4.12, one easily sees that Corollary 4.9 extends to the left, right p-equiintegrability and p-biequiintegrability.

Remark. If $\mathcal{A}$ is finite, the notions of left p-equiintegrability and right p-equiintegrability coincide (so the four notions of p-equiintegrability coincide). This can be deduced simply from Proposition 4.11 and the fact that s_e is central in this case. Consequently, in this case, the sequences $(p_k^{\perp}h_{n_k}p_k)$ and $(p_kh_{n_k}p_k^{\perp})$ in Corollary 4.12 converge to zero.

The following gives one more characterization of equiintegrability, which is quite useful in some context.

Proposition 4.13. *Let $0 < p < \infty$ and $\mathcal{A}$ be a von Neumann algebra with unit ball $B_{\mathcal{A}}$. Let K be a subset of $L_p(\mathcal{A})$. Then:*
i) K is left (resp. right) p-equiintegrable iff for every $\varepsilon > 0$ there exists $h_\varepsilon \in L_p(\mathcal{A})$ such that for every $h \in K$, the distance $d(h, h_\varepsilon.B_{\mathcal{A}})$ (resp. $d(h, B_{\mathcal{A}} \cdot h_\varepsilon)$) in $L_p(\mathcal{A})$ is majorized by ε.
ii) K is p-equiintegrable iff for every $\varepsilon > 0$ there exists $h_\varepsilon \in L_p(\mathcal{A})$ such that for every $h \in K$, $d(h, B_{\mathcal{A}}.h_\varepsilon + h_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon$.
iii) K is p-biequiintegrable iff for every $\varepsilon > 0$ there exists $h_\varepsilon \in L_p(\mathcal{A})$ such that for every $h \in K$, $d(h, B_{\mathcal{A}} \cdot h_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon$.
In the case where $\mathcal{A}$ is σ-finite, one can take elements h_ε of the form $M_\varepsilon h_0$, where M_ε is a positive real number and h_0 is a fixed positive element of $L_p(\mathcal{A})$ with full support ($s(h_0) = \mathbf{1}$).

Proof: We give the proof for left equiintegrable sets, and a non σ-finite von Neumann algebra; the other cases can be treated similarly. The sufficiency of the condition is clear since for every $h_0 \in L_p(\mathcal{A})$, the set $h_0 \cdot B_{\mathcal{A}}$ is left p-equiintegrable. Conversely, assume that K is left p-equiintegrable. Suppose that for some $\varepsilon > 0$ and for every finite subset F of $L_p(\mathcal{A})$ there exists $h_F \in K$ such that $d(h_F, \sum_{f \in F} (f \cdot B_{\mathcal{A}})) > \varepsilon$. Let $\mathcal{F}$ be the net of finite subsets of $L_p(\mathcal{A})$, ordered by inclusion; let Φ be the filter of cofinal subsets of $\mathcal{F}$ (generated by the set of final sections $\Sigma_F = \{G \in \mathcal{F} \mid F \subset G\}$); let finally $\mathcal{U}$ be an ultrafilter containing Φ. By Proposition 4.11, the element $\tilde{h} = (h_F)^\bullet$ of $L_p(\mathcal{A})_{\mathcal{U}}$ belongs to $s_e \cdot L_p(\mathcal{A})$. Hence by Lemma 4.7, there exists $h_\varepsilon \in L_p(\mathcal{A})$ such that $d(\tilde{h}, \hat{h}_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon$. By Kaplansky's density theorem we have in fact $d(\tilde{h}, \hat{h}_\varepsilon \cdot B_{\mathcal{A}_{\mathcal{U}}}) < \varepsilon$. Consequently, the set $\{F \in \mathcal{F} \mid d(h_F, h_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon\}$ belongs to $\mathcal{U}$. Since the set $\{F \in \mathcal{F} \mid h_\varepsilon \in F\} = \Sigma_{\{h_\varepsilon\}}$ belongs to $\mathcal{U}$ too, there is $F \in \mathcal{F}$ such that $h_\varepsilon \in F$ and $d(h_F, h_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon$, which contradicts the choice of the h_F's. So in fact for every $\varepsilon > 0$ there exists $F_\varepsilon = \{h_1^{(\varepsilon)}, ..., h_n^{(\varepsilon)}\} \in \mathcal{F}$ such that $d(h, \sum_{i=1}^{n} h_i^{(\varepsilon)} \cdot B_{\mathcal{A}}) \leq \varepsilon$ for every $h \in K$. Let $h_\varepsilon = (\sum_{i=1}^n h_i^{(\varepsilon)} h_i^{(\varepsilon)*})^{1/2}$, then for every $i = 1, ..., n$ we have $h_i^{(\varepsilon)} = h_\varepsilon x_i$, for some $x_i \in B_{\mathcal{A}}$. Consequently, $\sum_{i=1}^{n} h_i^{(\varepsilon)} \cdot B_{\mathcal{A}} \subset (nh_\varepsilon) \cdot B_{\mathcal{A}}$, and $d(h, (nh_\varepsilon) \cdot B_{\mathcal{A}}) \leq \varepsilon$ for every $h \in K$. $\square$

Historical comments. i) In the case of commutative L_1-spaces, Corollary 4.8 was proved in [KP] (where it is not explicitly stated but is a key ingredient of the proof of the main result there). There are various extensions to the Banach lattice setting; a general statement

was given by L. Weis using ultrapower techniques [W].
ii) In the non-commutative case a subsequence splitting lemma similar to Corollary 4.8 was obtained in [S] for symmetric spaces of measurable operators $E(\mathcal{A},\tau)$ associated with an order continuous rearrangement invariant space E and a von Neumann algebra $\mathcal{A}$ equipped with a finite trace τ (see Lemma 1.1 and Proposition 2.2 of [S]). Randrianantoanina proved Corollary 4.8 for symmetric spaces of measurable operators $E(\mathcal{A},\tau)$ when τ is semi-finite ([R1]) and for general non-commutative L_p-spaces ([R3]).
iii) In the case of finite and σ-finite von Neumann algebras Proposition 4.13 goes back to [HRS].

Application: weakly relatively compact sets in $\mathcal{A}_*$

The 1-equiintegrable sets coincide with the weakly relatively compact sets. Proposition 4.13 can be used to give a new proof of some well known results of C. A. Akeman ([A]):

Theorem 4.14. *Let K be a bounded subset of the predual $\mathcal{A}_*$ of a von Neumann algebra $\mathcal{A}$. The following assertions are equivalent:*
i) K is weakly relatively compact;
ii) For every sequence (p_n) of pairwise disjoint projections in $\mathcal{A}$, $\lim\limits_{n\to\infty}\sup\limits_{\varphi\in K}|\varphi(p_n)|=0$;
iii) K is 1-equiintegrable;
iv) There exists $\psi_0\in\mathcal{A}_*^+$ such that $\sup\limits_{\varphi\in K}|\varphi(a)|\to 0$ when $\psi_0(aa^*+a^*a)\to 0$, $a\in B_{\mathcal{A}}$.

Proof: The new ingredient will be the proof of (iii) $\Longrightarrow$ (iv); the proofs of the other implications are standard.

(i) $\Longrightarrow$ (ii): Let (p_n) be a disjoint sequence of projections in $\mathcal{A}$: they generate an abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$. Let ρ be the restriction map $\mathcal{A}_*\to\mathcal{B}_*$, $\varphi\mapsto\varphi_{|\mathcal{B}}$. Then $\rho(K)$ is weakly relatively compact in $\mathcal{B}_*\sim\ell_1$, and consequently (by the commutative result, i. e. Dunford-Pettis Theorem, see [D], p. 93), $\sup\{|f(p_n)|\mid f\in\rho(K)\}\to 0$, i.e. $\sup\{|\varphi(p_n)|\mid\varphi\in K\}\to 0$.

(ii) $\Longrightarrow$ (iii): Assume that for some disjoint sequence (p_n) of projections in $\mathcal{A}$ and some sequence (φ_n) in K we have $\inf\limits_n\|p_n\varphi_np_n\|=\delta>0$. For every n we have $\|p_n\varphi_np_n\|=\langle p_n\varphi_np_n,u_n\rangle=\varphi_n(p_nu_np_n)$ for some partial isometry u_n in $\mathcal{A}$. Let $p_nu_np_n=a_n+ib_n$ be the decomposition into real and imaginary parts. Then a_n,b_n belong to the unit ball of $p_n\mathcal{A}p_n$. Thus one of the sets $N_1=\{n\geq 1\mid|\varphi_n(a_n)|\geq\delta/2\}$ and $N_2=\{n\geq 1\mid|\varphi_n(b_n)|\geq\delta/2\}$ is infinite. Suppose w.l.o.g. that $|\varphi_n(a_n)|\geq\delta/2$ for every $n\geq 1$. The von Neumann algebra $\mathcal{C}$ generated by the a_n's is commutative (since they are hermitian and disjoint). The image $\rho(K)$ of the set K by the restriction map $\rho:\mathcal{A}_*\to\mathcal{C}_*$ still verifies (ii). However, in the commutative case, it is clear that (ii) implies (iii). Therefore, $\langle\varphi_n,a_n\rangle=\langle\rho(\varphi_n),a_n\rangle\to 0$ when $n\to\infty$, a contradiction.

(iii) $\Longrightarrow$ (iv): if K is 1-equiintegrable, then by Prop. 4.13 for every $n\geq 1$ there is $\varphi_n\in\mathcal{A}_*^+$ such that for every $\varphi\in K$ there exist $x,y\in B_{\mathcal{A}}$ such that $\|\varphi-(x\cdot\varphi_n+\varphi_n\cdot y)\|<1/n$. If $a\in B_{\mathcal{A}}$ we have then:

$$\begin{aligned}|\varphi(a)|&\leq\frac{1}{n}+|\langle x\cdot\varphi_n+\varphi_n\cdot y,a\rangle|\leq\frac{1}{n}+|\varphi_n(ax)|+|\varphi_n(ya)|\\&\leq\frac{1}{n}+\varphi_n(aa^*)^{1/2}\varphi_n(x^*x)^{1/2}+\varphi_n(a^*a)^{1/2}\varphi_n(yy^*)^{1/2}\\&\leq\frac{1}{n}+(2\|\varphi_n\|)^{1/2}(\varphi_n(aa^*)+\varphi_n(a^*a))^{1/2}\end{aligned}$$

Set $\psi_0 = \sum_{n\geq 1} 2^{-n} \|\varphi_n\|^{-1} \varphi_n$. Then if $\psi_0(aa^* + a^*a) < 2^{-n-1} \|\varphi_n\|^{-2} n^{-2}$, we obtain $|\varphi(a)| \leq 2/n$ for every $\varphi \in K$, and thus prove (iv).

(iv) $\Longrightarrow$ (i): Let $f \in \mathcal{A}^*$ be any w*-limit point of K. Then $|f(a)| \to 0$ when $\psi_0(aa^* + a^*a) \to 0$, $a \in B_{\mathcal{A}}$. Consequently, the linear functional f is strong*-continuous on bounded sets of $\mathcal{A}$, so it is w*-continuous ([T], Theorem II.2.6) and thus belongs to $\mathcal{A}_*$. ∎

Remark. The equivalence of conditions (i) and (iii) in Theorem 4.14 was also obtained in [HRS] for finite von Neumann algebras.

Another application of Proposition 4.13 is the following well known result of H. Jarchow [Ja]:

Theorem 4.15. *Every reflexive subspace of the predual of a von Neumann algebra is super-reflexive.*

Proof: Since a Banach space is reflexive iff its unit ball is weakly compact (or, equivalently, weakly relatively compact), then by Theorem 4.14, a closed subspace of a predual of von Neumann algebra is reflexive iff its unit ball is 1-equiintegrable. Let now X be a reflexive subspace of the predual $\mathcal{A}_*$. It is super-reflexive iff all its ultrapowers are reflexive. Such an ultrapower $X_{\mathcal{U}}$ is a closed subspace of $(\mathcal{A}_*)_{\mathcal{U}}$ which we identify with $\mathcal{A}_*$. Let $\varepsilon > 0$ and $\varphi_\varepsilon \in \mathcal{A}_*$ such that $d(\varphi, B_{\mathcal{A}} \cdot \varphi_\varepsilon + \varphi_\varepsilon \cdot B_{\mathcal{A}}) < \varepsilon$ for every $\varphi \in B_X$. Then clearly $d(\tilde{\varphi}, B_{\mathcal{A}_{\mathcal{U}}} \cdot \hat{\varphi}_\varepsilon + \hat{\varphi}_\varepsilon \cdot B_{\mathcal{A}_{\mathcal{U}}}) \leq \varepsilon$ for every $\tilde{\varphi} \in B_{X_{\mathcal{U}}}$, where $\hat{\varphi}_\varepsilon$ is the canonical image of φ_ε in $(\mathcal{A}_*)_{\mathcal{U}}$. Thus $d(\tilde{\varphi}, B_{\mathcal{A}} \cdot \hat{\varphi}_\varepsilon + \varphi_\varepsilon \cdot B_{\mathcal{A}}) \leq \varepsilon$ for every $\tilde{\varphi} \in B_{X_{\mathcal{U}}}$, and so $B_{X_{\mathcal{U}}}$ is 1-equiintegrable, hence weakly relatively compact. ∎

Remark. It is easy to see that if ψ_0 is the "control measure" for B_X given by Akemann's condition (iv) in Theorem 4.14, then $\widehat{\psi_0}$ is a control measure for $B_{X_{\mathcal{U}}}$ (in virtue of the strong*-density of $B_{\mathcal{A}_{\mathcal{U}}}$ in $B_{\mathcal{A}}$).

5. Subspaces containing ℓ_p

The following is the main result of this section. It gives several characterizations of the subspaces of $L_p(\mathcal{A})$ which contain ℓ_p.

Theorem 5.1. *Let $0 < p < \infty$, $p \neq 2$ and $\mathcal{A}$ be a von Neumann algebra. Let $X \subset L_p(\mathcal{A})$ be a closed subspace. The following statements are equivalent:*
i) X contains an almost disjoint normalized sequence.
ii) X contains a basic sequence asymptotically 1-equivalent to the ℓ_p-basis (and, if $1 \leq p < \infty$, spanning an almost 1-complemented subspace of $L_p(\mathcal{A})$).
iii) X contains a subspace isomorphic to ℓ_p.
iv) X contains uniformly the spaces ℓ_p^n, $n \geq 1$.
v) For some $q \in (0,p)$, (or equivalently, for every $0 < q < p$) and for every $h \in L_r(\mathcal{A})$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, the restriction $T_{h,p,q}\big|_X$ is not an isomorphism, where:

$$T_{h,p,q} : L_p(\mathcal{A}) \longrightarrow L_q(\mathcal{A}) \oplus L_q(\mathcal{A}),\ x \mapsto (xh, hx)$$

(If $\mathcal{A}$ is σ-finite and h_0 is an element of $L_r(\mathcal{A})_+$ with full support, it is sufficient to test this condition on $T_{h_0,p,q}$).
vi) If in addition $0 < p < 2$: the unit ball of X is not p-equiintegrable.

To prove the last equivalence in Theorem 5.1, we shall need the following lemma. As usual, we denote by (ε_n) a sequence of independent Bernouilli variables (random signs) and by $\mathbf{E}_\varepsilon$ the corresponding expectation.

Lemma 5.2. *Let $0 < p < 2$ and K be a p-equiintegrable sequence in $L_p(\mathcal{A})$. Then*

$$\lim_{n\to\infty} n^{-1/p} \sup \{ \mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i h_i \| \mid h_1, ..., h_n \in K \} = 0$$

Proof: For every $\alpha > 0$, we can find by Proposition 4.13 an element $h_0 \in L_p(\mathcal{A})_+$, and for every $h \in K$, elements x, y in the unit ball of $\mathcal{A}$ such that $\|h - xh_0 - h_0 y\| < \alpha$. Given $h_1, ..., h_n \in K$ let $x_1, y_1, ..., x_n, y_n \in B_{\mathcal{A}}$ such that $\|h_i - x_i h_0 - h_0 y_i\| < \alpha$, $i = 1, ..., n$. Let $r > 0$ such that $1/p = 1/2 + 1/r$. We have:

$$\mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i x_i h_0 \|_p \leq \mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i x_i h_0^{p/2} \|_2 \| h_0^{p/r} \|_r = \|h_0\|_p^{p/r} (\sum_{i=1}^n \| x_i h_0^{p/2} \|_2^2)^{1/2} \leq n^{1/2} \, \|h_0\|_p$$

and similarly,

$$\mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i h_0 y_i \|_p \leq n^{1/2} \, \|h_0\|_p$$

Since $L_p(\mathcal{A})$ is of type p with constant 1 (this follows from interpolation if $1 < p < 2$ and from the p-norm inequality if $0 < p < 1$), we have

$$\mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i (h_i - x_i h_0 - h_0 y_i) \|_p \leq (\sum_{i=1}^n \| h_i - x_i h_0 - h_0 y_i \|_p^p)^{1/p} \leq n^{1/p} \alpha$$

Combining the preceding inequalities, we deduce

$$\limsup_{n\to\infty} n^{-1/p} \sup \{ \mathbf{E}_\varepsilon \, \| \sum_{i=1}^n \varepsilon_i h_i \| \mid h_1, ..., h_n \in K \} \leq \alpha$$

which proves the lemma. □

Proof of Theorem 5.1. The implication (i) $\Longrightarrow$ (ii) is the implication (iii)$\Longrightarrow$ (iv) in the Prop. 2.11; the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are trivial. The implication (iv) $\Longrightarrow$ (i) is a special case of Theorem 3.1.

(i) $\Longrightarrow$ (v): if (a_n) is a normalized disjoint sequence, $a_n h \to 0$ and $h a_n \to 0$ for every $h \in L_r$. For $s_n h \to 0$ (resp. $h s_n \to 0$) in L_r for every disjoint sequence (s_n) of projections (since the set $\{h\}$ is p-biequiintegrable).

(v) $\Longrightarrow$ (i): let $\mathcal{F}$ be the set of finite subsets of $L_r(\mathcal{A})$, ordered by inclusion. Let Φ be the set of cofinal subsets of $\mathcal{F}$ and $\mathcal{U}$ an ultrafilter on $\mathcal{F}$ containing Φ (see the proof of Proposition 4.13). Note that the ultrafilter $\mathcal{U}$ is necessarily countably incomplete: so we can find a family (ε_F) of strictly positive real numbers such that $\lim_{F,\mathcal{U}} \varepsilon_F = 0$. By (v) we can choose for every $F \in \mathcal{F}$ an element $x_F \in X$, with $\|x_F\|_p = 1$, such that $\|x_F h\| < \varepsilon_F$ and $\|h x_F\| < \varepsilon_F$ for every $h \in F$ (apply hypothesis (v) to $h_F = (\sum_{h\in F} (|h|^2 + |h^*|^2))^{1/2}$). It follows that the family (x_F) defines an element ξ of $X_{\mathcal{U}}$ (hence of $L_p(\mathcal{A})_{\mathcal{U}}$) which verifies $\hat{h}\xi = 0 = \xi \hat{h}$ for every

$h \in L_r(\mathcal{A})$. Consequently, ξ is disjoint from $L_p(\mathcal{A})$, and so by Theorem 2.7 we can extract from the family (x_F) an almost disjoint sequence. Thus we get (i). In the case where $\mathcal{A}$ is σ-finite and $h_0 \in L_r(\mathcal{A})_+$ with full support such that $T_{h_0,p,q}$ is not an isomorphism, we choose a normalized sequence x_n in X such that $\|x_n h_0\| \to 0$ and $\|h_0 x_n\| \to 0$, and consider the element ξ defined by the sequence (x_n) in some ultrapower $X_{\mathcal{U}}$ (associated with a free ultrafilter over $\mathbb{N}$). Then ξ is disjoint from h_0 and consequently from $L_r(\mathcal{A})$ (since h_0 has full support), and finally from $L_p(\mathcal{A})$.

(i) $\Longrightarrow$ (vi): This is clear since a disjoint normalized sequence is not p-equiintegrable (nor is an almost disjoint normalized sequence).

(vi) $\Longrightarrow$ (i): if the unit ball of X is not p-equiintegrable, we can find a sequence (h_n) of normalized elements of X and a disjoint sequence (p_n) of projections of $\mathcal{A}$ such that $\|p_n h_n p_n\|_p > \delta > 0$ for every $n \geq 1$. By the Subsequence Splitting Lemma, we may suppose that $h_n = h'_n + h''_n$, where (h'_n) is p-equiintegrable and (h''_n) is disjoint. We have $p_n h'_n p_n \to 0$, so we may suppose that $\|p_n h''_n p_n\| > \delta$, and consequently $\|h''_n\| > \delta$ for every $n \geq 1$. Using Lemma 5.2, we can construct inductively a sequence $I_1 < ... < I_n < ...$ of disjoint intervals of $\mathbb{N}$ and a sequence of signs (ε_i) such that $|I_n|^{-1/p} \| \sum_{i \in I_n} \varepsilon_i h'_i \| < 2^{-n}$ for every $n \geq 1$. Let

$$a'_n = |I_n|^{-1/p} \sum_{i \in I_n} \varepsilon_i h'_i, \quad a''_n = |I_n|^{-1/p} \sum_{i \in I_n} \varepsilon_i h''_i \quad \text{and} \quad a_n = a'_n + a''_n$$

Then $(a_n) \subset X$, (a''_n) is equivalent to the ℓ_p-basis, and by a standard perturbation argument, $(a_n)_{n \geq n_0}$ is also equivalent to the ℓ_p-basis for sufficiently large n_0. □

The equivalence between (i) and (vi) in Theorem 5.1 can be extended to sequences in the following way.

Proposition 5.3. *Let $0 < p < 2$, and $(h_n) \subset L_p(\mathcal{A})$ be a bounded sequence. If $1 < p < 2$, suppose in addition that (h_n) is unconditional. Then the following assertions are equivalent:*
i) (h_n) is not p-equiintegrable;
ii) (h_n) contains a subsequence equivalent to the ℓ_p-basis.

Proof: That (ii) $\Longrightarrow$ (i) is a consequence of Lemma 5.2. The converse implication can be proved using arguments similar to those used in the semi-finite case by [HRS] for the case $1 \leq p < 2$ or [SX] for the case $p \leq 1$. We sketch these arguments for the convenience of the reader (with a modified, somewhat shortened proof in the case $0 < p < 1$). Assuming (i), we can choose, by Corollary 4.9, a subsequence of (h_n) (for simplicity of notation, we shall assume that it is (h_n) itself), and a disjoint sequence of projections (e_n) such that $\|e_n h_n e_n\| > \delta > 0$ for every $n \geq 1$.

a) The case $1 < p < 2$.

For every finite sequence (λ_n) of scalars, since the projections e_j are pairwise disjoint and $p \geq 1$, we have:

$$\mathbf{E}_\varepsilon \| \sum_n \varepsilon_n \lambda_n h_n \|^p \geq \mathbf{E}_\varepsilon \sum_{j=1}^{\infty} \|e_j (\sum_n \varepsilon_n \lambda_n h_n) e_j\|^p = \sum_{j=1}^{\infty} \mathbf{E}_\varepsilon \|e_j (\sum_n \varepsilon_n \lambda_n h_n) e_j\|^p$$
$$\geq \sum_{j=1}^{\infty} \|\lambda_j e_j h_j e_j\|^p \geq \delta^p \sum_j |\lambda_j|^p$$

Thus by the unconditionality of the sequence (h_n), we deduce that $\|\sum_n \lambda_n h_n\|^p \geq c\sum_n |\lambda_n|^p$ for some $c>0$. The converse inequality follows from the type p property of $L_p(\mathcal{a})$.
b) The case $p \leq 1$.

Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$; let $\tilde{h}$ be the element of $L_p(\mathcal{a})_{\mathcal{U}} = L_p(\mathcal{A})$ represented by the sequence (h_n), and for each m, let $\hat{e}_m$ be the canonical image of e_m in $\mathcal{a}_{\mathcal{U}}$. We have $\lim_{m\to\infty}\|\hat{e}_m\tilde{h}\| = 0$, since the projections $\hat{e}_m$ are pairwise disjoint. Similarly $\lim_{m\to\infty}\|\tilde{h}\hat{e}_m\| = 0$. Let m_k be such that $\|\hat{e}_m\tilde{h}\| + \|\tilde{h}\hat{e}_m\| < \varepsilon \cdot 2^{-k-1}$ for every $m \geq m_k$. On the other hand, $\lim_{n\to\infty}\|e_n h_m\| = 0 = \lim_{n\to\infty}\|h_m e_n\|$ for every $m \in \mathbb{N}$. So we can define inductively an increasing sequence (n_k), with $n_k \geq m_k$ for all k, in the following way: $n_1 = m_1$, and $n_{k+1} \geq \max(n_k+1, m_{k+1})$ is chosen such that for every $j = 1, ..., k$:

$$
\begin{aligned}
&i)\ \max(\|e_{n_{k+1}}h_{n_j}\|, \|h_{n_j}e_{n_{k+1}}\|) \leq \varepsilon 2^{-k-1}\\
&ii)\ \max(\|h_{n_{k+1}}e_{n_j}\|, \|e_{n_j}h_{n_{k+1}}\|) \leq \varepsilon 2^{-j}
\end{aligned}
$$

Then $\max(\|e_{n_j}h_{n_k}\|, \|h_{n_k}e_{n_j}\|) \leq \varepsilon 2^{-j}$ for every $j \neq k$. Set $e = \sum_{j\geq 1} e_{n_j}$ and $K = \sup\|h_n\|$. We have for every $(\lambda_j) \in \ell_p$:

$$
\begin{aligned}
K^p\sum_{j\geq 1}|\lambda_j|^p &\geq \|\sum_{j\geq 1}\lambda_j h_{n_j}\|^p \geq \|e(\sum_{j\geq 1}\lambda_j h_{n_j})e\|^p\\
&\geq \|\sum_{j\geq 1}\lambda_j e_{n_j}h_{n_j}e_{n_j}\|^p - \sum_{j\geq 1}\sum_{k\neq j}\|\lambda_j e_{n_k}h_{n_j}e\|^p - \sum_{j\geq 1}\sum_{k\neq j}\|\lambda_j e_{n_j}h_{n_j}e_{n_k}\|^p\\
&\geq (\delta^p - 2C^p\varepsilon^p)\sum_{j\geq 1}|\lambda_j|^p
\end{aligned}
$$

where $C^p = \sum_k 2^{-kp}$. Hence if $\varepsilon < 2^{-1/p}(\delta/C)$, the sequence (h_{n_j}) is equivalent to the ℓ_p-basis. $\square$

Application: Kadec-Pełczyński dichotomy for non-commutative L_p-spaces

Proof of Theorem 0.2. Theorem 0.2 follows from the special case $q=2$ of Theorem 5.1. Note that if for some $h \in L_r(\mathcal{a})$ the map $T_{h,p,2} : L_p(\mathcal{a}) \longrightarrow L_2(\mathcal{a}) \oplus_2 L_2(\mathcal{a})$ restricts to an isomorphism $S = T_{h,p,2}\mid_X$, then X is isomorphic to the Hilbert space $S(X)$; if $P : L_2(\mathcal{a}) \oplus_2 L_2(\mathcal{a}) \to S(X)$ is the orthogonal projection, then $Q := S^{-1}PT_{h,p,2}$ is a projection from $L_p(\mathcal{a})$ onto X. $\square$

Another version of the Kadec-Pełczyński dichotomy, which is well known in the commutative case, deals with unconditional sequences.

Proposition 5.4. *Let $2 < p < \infty$ and $\mathcal{a}$ be a von Neumann algebra. Then every semi-normalized unconditional sequence in $L_p(\mathcal{a})$ either is equivalent to the ℓ_2-basis or has a subsequence which is asymptotically* 1*-equivalent to the ℓ_p-basis and spans a complemented subspace.*

Proof: Let (h_n) be a semi-normalized unconditional sequence in $L_p(\mathcal{a})$ (by semi-normalized we mean that $0 < \delta = \inf_n \|h_n\|_p \leq M = \sup\|h_n\|_p < \infty$). Let $T_{h,p,2} : L_p(\mathcal{a}) \longrightarrow L_2(\mathcal{a}) \oplus_2 L_2(\mathcal{a})$ be as before. If for some $h \in L_r(\mathcal{a})$ (with $1/r = 1/2 - 1/p$), $\inf\{\|T_{h,p,2}h_n\|_2 \mid n \geq$

$1\} = c > 0$, then (h_n) is equivalent to the ℓ_2-basis. Indeed, then for every finite sequence (λ_n) of scalars:

$$\sqrt{2}\,\|h\|_r\,\mathbf{E}_\varepsilon\|\sum_n \lambda_n\varepsilon_n h_n\|_p \geq \mathbf{E}_\varepsilon\|\sum_n \lambda_n\varepsilon_n T_{h,p,2}h_n\|_2$$
$$= (\sum_n |\lambda_n|^2\|T_{h,p,2}h_n\|^2)^{1/2} \geq c\,(\sum_n |\lambda_n|^2)^{1/2}$$

on the other hand, by the type 2 property of $L_p(\mathcal{A})$,

$$\mathbf{E}_\varepsilon\|\sum_n \lambda_n\varepsilon_n h_n\|_p \leq C(\sum_n |\lambda_n|^2\|h_n\|^2)^{1/2} \leq CM(\sum_n |\lambda_n|^2)^{1/2}$$

If at the contrary we have $\inf\{\|T_{h,p,2}h_n\|_2 \mid n \geq 1\} = 0$ for any $h \in L_r(\mathcal{A})$, then we can adapt the proof of (v)$\Longrightarrow$ (i) of Theorem 5.1 by choosing the x_F's in the sequence (h_n). Then we deduce that (h_n) has an almost disjoint subsequence. $\square$

Remark. The results of [HRS] concerning the Banach-Saks properties of non-commutative L_p-spaces associated with a finite von Neumann algebra can be extended to the present setting with the same proof (using our Lemma 5.2 in place of Lemma 3.1 of [HRS]). We refer to [HRS], Definition 5.5 for the definition of the various Banach-Saks properties (the terminology is not completely fixed and their Banach-Saks property is sometimes called "weak Banach-Saks", or "Banach-Saks-Rosenthal" property). Then $L_1(\mathcal{A})$ has the Banach-Saks property, $L_p(\mathcal{A})$ has the p-Banach-Saks property if $1 \leq p \leq 2$ and the 2-Banach-Saks property if $2 \leq p < \infty$; any p-equiintegrable weakly null sequence of $L_p(\mathcal{A})$, $1 < p < 2$, has a strong p-Banach-Saks subsequence, and a closed linear subspace of $L_p(\mathcal{A})$, $1 < p < 2$ has the strong p-Banach-Saks property iff it has no subspace isomorphic to ℓ_p.

Historical comments. i) The commutative forerunner of Theorem 5.1 is due to H. P. Rosenthal [Ro]. For finite von Neumann algebras, Theorem 5.1 was proved in [HRS] for $1 \leq p < 2$ and in [SX] for $p < 1$. The proofs in both papers use the notion of the p-modulus of uniform integrability, the definition of which involves the trace. An analogue of this modulus could be defined in the non-tracial, σ-finite case too, via a normal faithful state, but it would have less tractable properties (in particular with respect to conjugation and absolute value). For finite von Neumann algebras, some equivalences in Theorem 5.1 were obtained in [R1-2].

ii) Lemma 5.2 and Proposition 5.3 above are simply extensions to our context of the corresponding results for finite von Neumann algebras in [HRS].

iii) Theorem 0.2 and Proposition 5.4 were proved in the commutative context in the well-known paper of M. I. Kadec and A. Pełczyński [KaP]. These results were proved by Sukochev [S] in the case of a finite von Neumann algebra, and by N. Randrianantoanina [R2] in the semi-finite case (even in the more general setting of spaces $E(\mathcal{A}, \tau)$ associated with an order-continuous type 2 r.i. function space E).

6. Operator space version

This section is devoted to the analogues of Theorems 3.1 and 0.2 in the category of operator spaces. Our references for Operator Space Theory are [ER] and [P2]. Recall that an operator space E is a closed subspace of $B(H)$ for some Hilbert space H, equipped with

a natural sequence of matrix norms. Let M_n denote the space of complex $n \times n$ matrices and $M_n(E) = M_n \otimes E$ the space of $n \times n$ matrices with entries in E. As usual, $M_n(B(H))$ is identified with $B(\ell_2^n(H))$ and the matrix norm on $M_n(E)$ is the one induced by the natural inclusion of $M_n(E)$ into $M_n(B(H))$. An abstract characterization of operator spaces was given by Ruan (see [ER2]): a Banach space E equipped with a sequence $(\| \ \|_n)$ of norms on the $M_n(E)$ can be identified with an operator space iff the matricial norms $(\| \ \|_n)$ satisfy two simple conditions ("Ruan's axioms").

Now let E, F be two operator spaces; a linear map $T : E \to F$ is said to be completely bounded if

$$\|T\|_{cb} := \sup_{n \geq 1} \|\mathrm{id}_{M_n} \otimes T : M_n(E) \to M_n(F)\| < \infty$$

Then $\| \ \|_{cb}$ defines a norm on the space $CB(E, F)$ of completely bounded operators from E into F. An operator $T : E \to F$ is called a complete isomorphism if it is a linear bijection such that T and T^{-1} are completely bounded. Two operator spaces E, F are said to be completely isomorphic (resp. K-completely isomorphic) if there exists a complete isomorphism $T : E \to F$ (resp. with $\|T\|_{cb} \|T^{-1}\|_{cb} \leq K$). Similarly, a linear subspace F of E is said to be completely complemented (resp. K-completely complemented) in F if there is a completely bounded projection $P : E \to F$ (resp. with $\|P\|_{cb} \leq K$).

Now let $\mathcal{A}$ be a von Neumann algebra and $1 \leq p \leq \infty$. We will consider the natural operator space structure on $L_p(\mathcal{A})$ as introduced in [P1-2]. For $p = \infty$ a realization of $\mathcal{A}$ as a concrete von Neumann algebra, i.e. a unital w*-closed *-subalgebra of some $B(H)$, gives an operator space structure on $\mathcal{A} = L_\infty(\mathcal{A})$ (independent of the realization since *-isomorphisms are completely isometric). A standard operator space structure on the dual space $\mathcal{A}^*$ follows by Operator Spaces Theory ($M_n(\mathcal{A}^*)$ is identified with the space $CB(\mathcal{A}, M_n)$ and the corresponding sequence of matricial norms satisfies Ruan's axioms). A specific operator space structure on $L_1(\mathcal{A}) = \mathcal{A}_*$ is induced by the natural embedding of $\mathcal{A}_*$ into its bidual $\mathcal{A}^*$. In fact, as explained in [P2], §7 it is more convenient to consider $L_1(\mathcal{A})$ as the predual of the opposite von Neumann algebra $\mathcal{A}^{op}$, which is isometric (but not completely isomorphic) to $\mathcal{A}$, and to equip $L_1(\mathcal{A})$ with the operator space structure inherited from $(\mathcal{A}^{op})^*$. The main reason for this choice is that it insures that the equality $L_1(M_n \otimes \mathcal{A}) = S_1^n \widehat{\otimes} L_1(\mathcal{A})$ (operator space projective tensor product) holds true (see [Ju3], §3). Finally the operator space structure of $L_p(\mathcal{A})$ is obtained by complex interpolation, using the well known interpretation of $L_p(\mathcal{A})$ as interpolation space $(\mathcal{A}, L_1(\mathcal{A}))_{1/p}$ (see [Te2]).

We will need the following convenient characterization of the operator space structure of the subspaces of $L_p(\mathcal{A})$. Note that there is a natural *algebraic* identification of $L_p(M_n \otimes \mathcal{A})$ with $M_n(L_p(\mathcal{A}))$. Following [P1], if E is an operator space one sets $S_p^n[E] := (S_\infty^n[E], S_1^n[E])_{1/p} = (M_n[E], S_1^n \widehat{\otimes} E)_{1/p}$; by [P1], Cor 1.4 we have when (E_0, E_1) is a compatible interpolation couple: $S_p^n[(E_0, E_1)_{1/p}] = (M_n[E_0], S_1^n \widehat{\otimes} E_1)_{1/p}$. Consequently we have: $S_p^n[L_p(\mathcal{A})] = (M_n \otimes \mathcal{A}, L_1(M_n \otimes \mathcal{A}))_{1/p} = L_p(M_n \otimes \mathcal{A})$; in other words $S_p^n[\mathcal{A}]$ identifies with the linear space $M_n(L_p(\mathcal{A}))$ equipped with the norm of $L_p(M_n \otimes \mathcal{A})$. Note that if $\mathcal{A}$ is commutative, say $\mathcal{A} = L_p(\Omega, \mu)$ for some measure space (Ω, μ), it turns out that $S_p^n[L_p(\mathcal{A})] = L_p(\Omega, \mu; S_p^n)$, the space of p-integrable functions with values in S_p^n. Recall also that if F is a closed linear subspace of E, the norm on $S_p^n[F]$ is induced from that of $S_p^n[E]$. The norms on $S_p^n[E], n \geq 1$, completely determine the operator space structure of E in the following sense (see [P1], prop. 2.3):

Lemma 6.1. *Let E_1 and E_2 be two operator spaces. Then a linear map $T : E_1 \to E_2$ is completely bounded iff*

$$\sup_n \|\mathrm{id}_{S_p^n} \otimes T : S_p^n[E_1] \to S_p^n[E_2]\| < \infty$$

moreover in this case the supremum above is equal to $\|T\|_{cb}$.

The embedding results in sections 3 and 5 can be improved into results for the category of operator spaces. We first consider subspaces of $L_p(\mathcal{A})$ containing ℓ_p.

Theorem 6.2. *Let* $0 < p < \infty$, $p \neq 2$ *and* X *be a closed subspace of* $L_p(\mathcal{A})$. *If* X *contains uniformly the spaces* ℓ_p^n, $n \geq 1$ *as Banach spaces, then given any* $\varepsilon > 0$, X *contains a subspace* $(1+\varepsilon)$*-completely isomorphic to* ℓ_p *and* $(1+\varepsilon)$*-completely complemented in* $L_p(\mathcal{A})$.

As a corollary we immediately get the following operator space version of the Kadec-Pełczyński dichotomy:

Corollary 6.3. *Let* $2 < p < \infty$, *and* X *be a closed subspace of* $L_p(\mathcal{A})$. *Then either* X *is (Banach) isomorphic to a Hilbert space or* X *contains a subspace which is completely isomorphic to* ℓ_p *and completely complemented in* $L_p(\mathcal{A})$.

Remark. Corollary 6.3. has been known to M. Junge and the second author for a semifinite $\mathcal{A}$ (and also when $\mathcal{A}$ is a type III algebra of some particular form).

Now we turn to the operator space analogue of Theorem 3.1. In the following, given an operator space F, the spaces $\ell_p(F)$ and $\ell_p^n(F)$ are equipped with the natural operator space structure introduced in [P1] (via complex interpolation). More generally, if $(F_j)_{j\geq 1}$ is a sequence of operator spaces, the space $(\bigoplus_{j\geq 1} F_j)_p$ has also a natural operator space structure.

Theorem 6.4. *Let* $1 \leq p < \infty$, *and* X *be a closed subspace of* $L_p(\mathcal{A})$. *Let* $(F_j)_{j\geq 1}$ *be a sequence of finite dimensional operator spaces. Assume that there is a constant* K *such that for all* $n, j \geq 1$, X *contains a subspace* $Y_{j,n}$ *which is* K*-completely isomorphic to* $\ell_p^n(F_j)$.
i) Then for every $\varepsilon > 0$, X *contains a subspace* $(K+\varepsilon)$*-completely isomorphic to* $F = (\bigoplus_{j\geq 1} F_j)_p$.
ii) If in addition each $Y_{j,n}$ *is* C*-completely complemented in* $L_p(\mathcal{A})$ *then* X *contains a subspace* $(K+\varepsilon)$*-completely isomorphic to* F *and* $(CK+\varepsilon)$*-completely complemented in* $L_p(\mathcal{A})$.

Specializing to Schatten classes we get the following:

Corollary 6.5. *Let* $1 \leq p < \infty$, *and* X *be a closed subspace of* $L_p(\mathcal{A})$. *If* X *contains subspaces uniformly completely isomorphic to* S_p^n, $n \geq 1$, *(resp. and uniformly completely complemented in* $L_p(\mathcal{A})$*) then* X *contains a subspace completely isomorphic to* $K_p = (\bigoplus_{n\geq 1} S_p^n)_p$ *(resp. and completely complemented in* $L_p(\mathcal{A})$*).*

Remarks. i) Corollary 6.5 can be used to simplify some proofs in [JNRS].

ii) It is worth noting that contrary to Theorem 6.4, the assumption in Theorem 6.2 is only at the Banach space level! (So the latter cannot be considered as a special case of the former.)

The proofs of Theorems 6.2 and 6.4 are very similar and that of Theorem 6.2 is simpler, so we give only the proof of the latter.

Proof of Theorem 6.4. By the proof of Theorem 3.1, given a sequence (ε_j) of positive real numbers (the ε_j's being very small), there is a sequence (E_j) of finite dimensional subspaces

of X and a disjoint sequence (p_j) of projections of $\mathcal{A}$, such that E_j is K-isomorphic to F_j by some isomorphism $T_j : F_j \to E_j$, and such that

$$\forall j \geq 1, \forall h \in E_j, \qquad \|h - p_j h p_j\| \leq \varepsilon_j \|h\| \tag{$*$}$$

Define $T : F \to E = \sum_j E_j$, $x = (x_j) \mapsto Tx = \sum_j T_j x_j$. Then T is an isomorphism, see the proof of Theorem 3.1.

Reexamining that proof, we see that, under the present hypothesis, each of the spaces E_j constructed there is in fact K-completely isomorphic to the corresponding F_j. More precisely, T_j can be defined so that

$$\left\|T_j^{-1}\right\|_{cb} \leq 1 \text{ and } \|T_j\|_{cb} \leq K$$

Indeed, keeping the notations used in the proofs of Lemma 3.2 and Theorem 3.1, we have that T_j is some $S_{j,i,n}$ at the beginning of the proof of Theorem 3.1 and $E_j = S_{j,i,n}(F_j)$. However, $S_{j,i,n} = T_{j,n}\, I_i$, where $I_i : F_j \to \ell_p^n(F_j)$ is the natural embedding of F_j into the i-th coordinate of $\ell_p^n(F_j)$, i.e. $I_i(x) = e_i \otimes x$, and where $T_{j,n} : \ell_p^n(F_j) \to X$ is the embedding given by Lemma 3.2. More precisely, by the discussion at the end of the proof of Lemma 3.2, for every n, j there are an integer N and a linear map $S_n : \ell_p^n \to \ell_p^N$ which satisfy the following: firstly, S_n sends the basis vectors of ℓ_p^n into disjoint blocks of ℓ_p^N; secondly, $T_{j,n} = T^{j,N}\left(S_n \otimes \mathrm{Id}_{F_j}\right)$, where $T^{j,N} : \ell_p^N(F_j) \to X$ is a K-complete embedding whose existence is guaranteed by the assumption of Theorem 6.4. Therefore, we obtain that

$$S_{j,i,n} = T^{j,N}\left(S_n \otimes \mathrm{Id}_{F_j}\right) I_i\,.$$

Since both I_i and $S_n \otimes \mathrm{Id}_{F_j}$ are completely isometric embeddings, we deduce the desired assertion on T_j.

We shall show that T is now a complete isomorphism. To this end, by Lemma 6.1 we need only to consider

$$\mathrm{id}_{S_p^m} \otimes T : S_p^m[F] \to S_p^m[E], \qquad m \geq 1$$

Fix $m \geq 1$ and let $\tilde{p}_j = \mathrm{id}_{\ell_2^m} \otimes p_j$. Then $(\tilde{p}_j)$ is a disjoint sequence of projections in $M_n \otimes \mathcal{A}$. We claim that (with $d_j = \dim E_j$):

$$\forall m, j \geq 1, \forall h \in S_p^m[E_j], \qquad \|h - \tilde{p}_j h \tilde{p}_j\|_{S_p^m[E_j]} \leq d_j \varepsilon_j \|h\|_{S_p^m[E_j]} \tag{$\dagger$}$$

(Note that the constant on the right hand side does not depend on m). Indeed, choose an Auerbach basis $(\xi_i)_{1 \leq i \leq d_j}$ in E_j and let $(\xi_i^*)_{1 \leq i \leq d_m}$ be the dual basis in the dual space E_j^*. For all $i, k = 1, ..., d_j$, $i \neq k$ we have $\langle \xi_i, \xi_i^* \rangle = \|\xi_i\| = \|\xi_i^*\| = 1$, and $\langle \xi_i, \xi_k^* \rangle = 0$. Every $h \in S_p^m[E_j]$ can be written as $h = \sum_{i=1}^{d_j} a_i \otimes \xi_i$, where the a_i, $i = 1, ..., d_j$ belong to S_p^m. Thus by $(*)$:

$$\|h - \tilde{p}_j h \tilde{p}_j\|_{S_p^m[E_j]} \leq d_j \varepsilon_j \sup_i \|a_i\|_{S_p^m}$$

Recall that any bounded functional on an operator space is automatically completely bounded, and that its cb-norm is equal to its norm. Hence

$$\|\mathrm{id}_{S_p^m} \otimes \xi_i^* : S_p^m[E_j] \to S_p^m\| = 1, \quad 1 \leq i \leq d_j$$

whence

$$\sup_i \|a_i\|_{S_p^m} \le \|h\|_{S_p^m[E_j]}$$

Combining the previous inequalities we obtain our claim (†). Now using (†) instead of (∗) and repeating the arguments in the proof of Theorem 3.1 with $\mathrm{id}_{S_p^m} \otimes T$ in place of T, we deduce that, if $d_j\varepsilon_j < 1$ for each j and $\displaystyle\sum_j \frac{d_j\varepsilon_j}{1-d_j\varepsilon_j} = \varepsilon < 1$,

$$\|\mathrm{id}_{S_p^m} \otimes T^{-1}\| \le (1-\varepsilon)^{-2}, \qquad \|\mathrm{id}_{S_p^m} \otimes T\| \le (1+\varepsilon)K$$

Since $m \ge 1$ is arbitrary, we obtain:

$$\|T^{-1}\|_{cb} \le (1-\varepsilon)^{-2}, \qquad \|T\|_{cb} \le (1+\varepsilon)K$$

This proves the part (i) of Theorem 6.4. The part (ii) can be proved similarly by combining the previous arguments with the proof of the part iii) of Theorem 3.1. ∎

Appendix: Equality case in non-commutative Clarkson inequality

Theorem A1. *Let $\mathcal{A}$ be a von Neumann algebra and $0 < p < \infty$, $p \ne 2$. Two elements x, y of $L_p(\mathcal{A})$ verify the equality:*

$$\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p)$$

if and only if they are disjoint.

This result was stated by H. Kosaki [Ko2] in the case $p > 2$, and proved by reduction to the equality case of an inequality valid in $L_{p/2}(\mathcal{A})_+$. We shall follow the same pattern, but the argument is different when $p < 2$. The equality case of this auxiliary inequality is given by the following:

Proposition A2. *Let $\mathcal{A}$ be a von Neumann algebra and $0 < r < \infty$, $r \ne 1$. Two positive elements a, b of $L_r(\mathcal{A})$ verify the equality:*

$$\mathrm{Tr}(a+b)^r = \mathrm{Tr}(a^r) + \mathrm{Tr}(b^r)$$

if and only if they are disjoint.

We first deduce Theorem A1 from Proposition A2. Let $r = p/2$ and $a = x^*x$, $b = y^*y$. Then:

$$\begin{aligned}\mathrm{Tr}(a^r) + \mathrm{Tr}(b^r) = \|x\|^p + \|y\|^p &= \frac{1}{2}(\|x+y\|^p + \|x-y\|^p)\\ &= \frac{1}{2}\mathrm{Tr}[(a+b+(x^*y+y^*x))^r + (a+b-(x^*y+y^*x))^r]\\ &\begin{cases} \le \mathrm{Tr}(a+b)^r \text{ if } 0 < r \le 1\\ \ge \mathrm{Tr}(a+b)^r \text{ if } r > 1\end{cases}\end{aligned}$$

where we have used the operator-concavity of the function $t \mapsto t^r$ if $0 < r \le 1$ (see [B], chap. V), and the convexity of the L_r-norm and of the function $t \mapsto t^r$ if $r \ge 1$. Note that the reverse inequalities are always true:

$$\mathrm{Tr}(a+b)^r \begin{cases} \le \mathrm{Tr}(a^r) + \mathrm{Tr}(b^r) \text{ if } 0 < r \le 1\\ \ge \mathrm{Tr}(a^r) + \mathrm{Tr}(b^r) \text{ if } r > 1\end{cases}$$

(see [Ko3] Lemma 3 in the first case and [Ko2] Lemma 3.3 in the second case). So we are in the equality case

$$\mathrm{Tr}(a+b)^r = \mathrm{Tr}(a^r) + \mathrm{Tr}(b^r)$$

and by Proposition A2 above this implies that a and b have disjoint supports. Since the support of a (resp. b) coincides with the right support of x (resp of y) this means that x and y have disjoint right supports. Replacing x, y by their conjugates, we see that the left supports of x and y are disjoint too, so $x \perp y$. ∎

Proof of Proposition A2: The case $r > 1$ was treated in [Ko2] Proposition 6.3, using a differentiation argument in the strictly convex Banach space $L_r(\mathcal{A})$. So we consider only the case $0 < r < 1$.

¿From $0 \leq a, b \leq a + b$ we infer the existence of $c, d \in \mathcal{A}$ with $\|c\| \leq 1$, $\|d\| \leq 1$ such that

$$a^{1/2} = c(a+b)^{1/2}, \quad b^{1/2} = d(a+b)^{1/2}$$

Hence

$$a = (a+b)^{1/2} c^* c (a+b)^{1/2} = c(a+b)c^*$$
$$b = (a+b)^{1/2} d^* d (a+b)^{1/2} = d(a+b)d^*$$

Note that we can choose c, d such that $c^*c + d^*d = s(a+b)$. Since $0 \leq r \leq 1$ we have by Hansen's inequality (see [Han] for bounded operators, and [Ko2] Lemma 3.6 for operators in $L_p(\mathcal{A})$):

$$a^r = (c(a+b)c^*)^r \geq c(a+b)^r c^*$$
$$b^r = (d(a+b)d^*)^r \geq d(a+b)^r d^*$$

Hence

$$\begin{aligned} \mathrm{Tr}(a^r + b^r) &\geq \mathrm{Tr}(c(a+b)^r c^*) + \mathrm{Tr}(d(a+b)^r d^*) \\ &= \mathrm{Tr}((a+b)^r (c^*c + d^*d)) \\ &= \mathrm{Tr}(a+b)^r = \mathrm{Tr}(a^r + b^r) \end{aligned}$$

so the inequalities above become all equalities; then we have:

$$a^r = c(a+b)^r c^*$$
$$b^r = d(a+b)^r d^*$$

(since the differences are positive and of zero trace). We distinguish now two cases:

<u>Case 1</u>: $r \leq 1/2$. Since $2r \leq 1$ we may use Hansen's inequality again:

$$a^r = c((a+b)^{1/2})^{2r} c^* \leq (c(a+b)^{1/2} c^*)^{2r} \leq (a^{1/2})^{2r} = a^r \tag{$*$}$$

where the last inequality follows from the inequality $c(a+b)^{1/2}c^* \leq (c(a+b)c^*)^{1/2} = a^{1/2}$ (Hansen's inequality) and $2r \leq 1$. Therefore, the inequalities in $(*)$ above are equalities:

$$a^r = (c(a+b)^{1/2} c^*)^{2r}$$

whence

$$a^{1/2} = c(a+b)^{1/2} c^*$$

equivalently:

$$a^{1/2} = c a^{1/2} = a^{1/2} c^*$$

which implies in particular that $s(a) \leq r(c)$ and that $a = cac^*$. Recalling that $a = c(a+b)c^*$, we see that $cbc^* = 0$, hence $r(c) \perp s(b)$. So finally $s(a) \perp s(b)$, which ends the proof of case 1.

<u>Case 2</u>: $1/2 < r < 1$. By the equalities $a^r = c(a+b)^r c^*$ and $a^{1/2} = c(a+b)^{1/2}$ we have

$$a^r = a^{1/2}(a+b)^{r-1/2}c^*$$

whence $a^{r-1/2} = s(a)(a+b)^{r-1/2}c^*$, and so:

$$a^{2r-1} = c(a+b)^{r-1/2}s(a)(a+b)^{r-1/2}c^* \leq c(a+b)^{2r-1}c^*$$

but since $2r-1 \leq 1$, we may use Hansen's inequality again:

$$c(a+b)^{2r-1}c^* \leq (c(a+b)c^*)^{2r-1} = a^{2r-1}$$

and thus we obtain the equality:

$$a^{2r-1} = c(a+b)^{2r-1}c^*$$

If $2r-1 \leq 1/2$, i.e. $r \leq 3/4$, then as in Case 1, we deduce that $a^{1/2} = c(a+b)^{1/2}c^*$ and then $s(a) \perp s(b)$. If not, we iterate the procedure. Define the sequence (r_n) by $r_0 = r$, $r_{n+1} = 2r_n - 1$. The interval $(1/2, 1]$ contains finitely many points of this sequence (which converges to $-\infty$). Let N be the first integer such that $r_N \leq 1/2$. We have $0 < r_N \leq 1/2$ and $1/2 < r_n < 1$ for $n = 0, ..., N-1$. So we have inductively

$$a^{r_n} = (c(a+b)c^*)^{r_n}$$

for $n = 0, ..., N$. For $n = N$ this equality implies $a^{1/2} = c(a+b)^{1/2}c^*$ and finally that $s(a) \perp s(b)$. $\square$

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